An invitation to port-Hamiltonian systems

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Summary

Lagrangian and Hamiltonian form of a bar under axial loading

Port-Hamiltonian formalism

To go further: the $\mathbb{R}^d$ case

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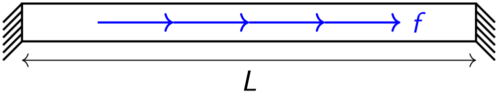
To go further: the $\mathbb{R}^d$ case

A bar under static tension-compression

$$-k\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = f, \qquad x \in [0, L],$$
$$q(0) = q(L) = 0, \qquad \text{bcs.}$$



$k := EA$ is the axial rigidity and f an external load.
Where does this come from?

1. use **Newton's law** in an infinitesimal portion $\mathrm{d}x$
2. or use the

Virtual work principle

For a structure at the equilibrium, the internal virtual work done by internal stresses equals the external virtual work done by external forces $\delta U = \delta W_{\text{ext}}$.

The energetic viewpoint

The elastic energy and external work are given by

$$U := \frac{1}{2}\int_0^L k\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2 \mathrm{d}x, \qquad W_{\text{ext}} = \int_0^L f \cdot q \,\mathrm{d}x.$$

The virtual work principle implies

$$\int_0^L k\frac{\mathrm{d}\delta q}{\mathrm{d}x}\frac{\mathrm{d}q}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L \delta q \cdot f \,\mathrm{d}x, \qquad \forall \delta q \quad \text{such that} \quad \delta q(0) = \delta q(L) = 0.$$

This is a **weak formulation** and it is **more general** than the previous ODE:

- if the solution is smooth enough, we retrieve $-k\dfrac{\mathrm{d}^2 q}{\mathrm{d}x^2} = f$,
- otherwise this formulation makes sense in less regular spaces.

Longitudinal waves

Now the inertial effects are included in the problem

$$\rho \partial_{tt} q - \partial_x (k \partial_x q) = f, \qquad \rho \text{ is the density per unit length.}$$

To obtain the equation one can use Newton's law or

Hamilton's principle

Among admissible motions, the actual motion of a system is such that the value of the integral

$$S = \int_{t_1}^{t_2} (T - U + W_{\mathrm{ext}}) \mathrm{d}t, \qquad \text{where} \quad T = \int_0^L \rho (\partial_t q)^2 \mathrm{d}x,$$

is minimized.

Euler Lagrange equations

Euler Lagrange equations

The minimization of S leads to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{q}} - \frac{\delta L}{\delta q} = f,$$

where the Lagrangian is defined as $L := T - U$.

The variational derivative $\frac{\delta J}{\delta f}$ of a functional $J(f) = \int_\Omega j(f(x), f'(x))\,\mathrm{d}x$ is defined as

$$\int_0^L \frac{\delta J}{\delta f} \cdot \delta f \,\mathrm{d}x := \lim_{\varepsilon \to 0} \frac{J(f + \varepsilon \delta f) - J(f)}{\varepsilon}, \qquad \varepsilon \in \mathbb{R}.$$

The Hamiltonian formalism

The Hamiltonian (total energy) is the Legendre transform of the Lagrangian

$$H(q,p) = \int_0^L p\dot{q}\,\mathrm{d}x - L(q,\dot{q}), \qquad \text{where} \quad p := \delta_{\dot{q}} L \quad \text{is the conjugate momentum.}$$

Then the Euler-Lagrange equations are equivalent to

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \delta_q H \\ \delta_p H \end{bmatrix} + \begin{bmatrix} f \\ 0 \end{bmatrix}, \qquad \begin{aligned} \delta_q H &= -\partial_x(k\partial_x q), \\ \delta_p H &= p/\rho. \end{aligned}$$

Finite element discretization

The **discrete Lagrangian** form reads

$$\mathbf{M}_\rho \ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}, \qquad \text{where} \quad [\mathbf{M}_\rho]_{ij} = \int_0^L \rho \, \phi_i \phi_j \mathrm{d}x.$$

The **discrete Hamiltonian** equations are given by

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{M}_\rho^{-1} \end{bmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} + \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \qquad H = \frac{1}{2}\mathbf{p}^\top \mathbf{M}_\rho^{-1} \mathbf{p} + \frac{1}{2}\mathbf{q}^\top \mathbf{K}\mathbf{q}.$$

Remark: we can equivalently rewrite using the velocity

$$\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{v} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}, \qquad H = \frac{1}{2}\mathbf{v}^\top \mathbf{M}_\rho \mathbf{v} + \frac{1}{2}\mathbf{q}^\top \mathbf{K}\mathbf{q}.$$

Time integration in Lagrangian dynamics

For Lagrangian dynamics the most well known integrator is the Newmark scheme:

$$
\begin{aligned}
\mathbf{M}_\rho \mathbf{a}^{n+1} + \mathbf{K}\mathbf{q}^{n+1} &= 0, \\
\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} &= \gamma \mathbf{a}^{n+1} + (1-\gamma)\mathbf{a}^n, \\
\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} &= \mathbf{v}^n + \frac{\Delta t}{2}(2\beta \mathbf{a}^{n+1} + (1-2\beta)\mathbf{a}^n).
\end{aligned}
$$

Two common choices:

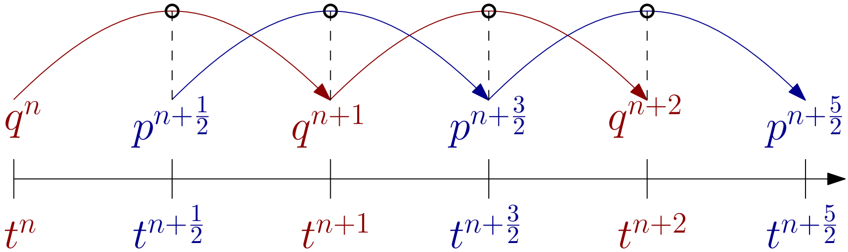
- $\gamma = \frac{1}{2}$, $\beta = 0$: Explicit Newmark (or Leapfrog scheme, or centered differences).
- $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$: Implicit Newmark.

Time integration in Hamiltonian dynamics

The **explicit Newmark** scheme is **equivalent** the **Störmer-Verlet** in Hamiltonian dynamics

$$\frac{\mathbf{p}^{n+\frac{1}{2}} - \mathbf{p}^{n-\frac{1}{2}}}{\Delta t} = -\mathbf{K}\mathbf{q}^n,$$

$$\mathbf{M}_\rho \frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} = \mathbf{p}^{n+\frac{1}{2}}.$$

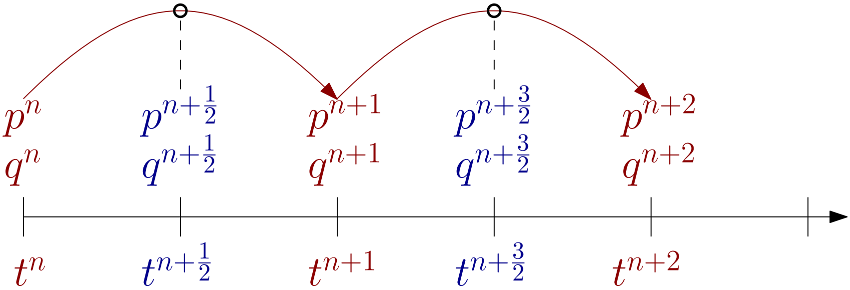


Time integration in Hamiltonian dynamics

The **implicit Newmark** scheme scheme is **equivalent** to the **implicit midpoint**

$$\frac{1}{\Delta t}\begin{bmatrix}\mathbf{I} & 0 \\ 0 & \mathbf{M}_\rho\end{bmatrix}\begin{pmatrix}\mathbf{p}^{n+1}-\mathbf{p}^n \\ \mathbf{q}^{n+1}-\mathbf{q}^n\end{pmatrix} = \begin{pmatrix}\mathbf{p}^{n+\frac{1}{2}} \\ -\mathbf{K}\mathbf{q}^{n+\frac{1}{2}}\end{pmatrix},$$

where $\mathbf{p}^{n+\frac{1}{2}} = \frac{\mathbf{p}^{n+1}+\mathbf{p}^n}{2}$, $\mathbf{q}^{n+\frac{1}{2}} = \frac{\mathbf{q}^{n+1}+\mathbf{q}^n}{2}$.



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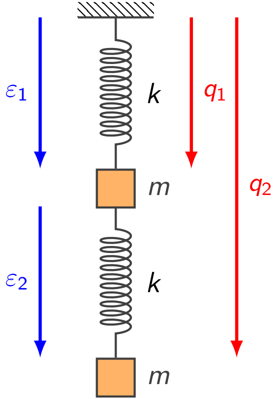
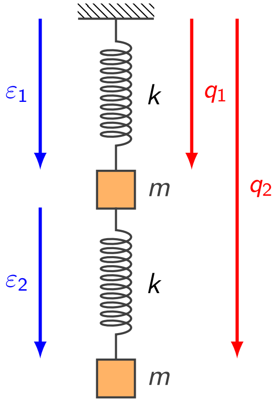
Hamiltonian and port-Hamiltonian formulation of a two dof oscillator[1]



[1]Schaft and B. M. Maschke, "Port-Hamiltonian Systems on Graphs".

Hamiltonian and port-Hamiltonian formulation of a two dof oscillator[1]



Canonical Hamiltonian formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} = \begin{bmatrix}0 & \mathbf{I}\\ -\mathbf{I} & 0\end{bmatrix}\begin{pmatrix}\partial_{\mathbf{q}} H\\ \partial_{\mathbf{p}} H\end{pmatrix}.$$

- $\mathbf{p} = (p_1 \quad p_2)^\top = (m\dot{q}_1 \quad m\dot{q}_2)^\top$ linear momenta;
- $\mathbf{q} = (q_1 \quad q_2)^\top$ position of the masses;
- $H = \frac{1}{2}k||\mathbf{D}\mathbf{q}||^2 + \frac{1}{2m}||\mathbf{p}||^2$, where $\mathbf{D} = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$.

Remark: notice that

$$U := \frac{1}{2}k||\mathbf{D}\mathbf{q}||^2 = \frac{1}{2}\mathbf{q}^\top\mathbf{K}\mathbf{q}, \qquad \mathbf{K} = k\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}.$$

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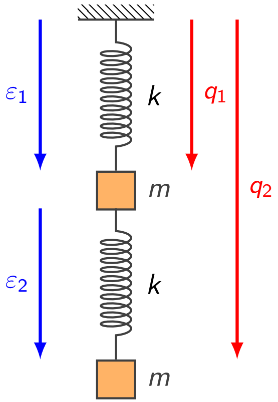
Hamiltonian and port-Hamiltonian formulation of a two dof oscillator[1]



Interconnection based formulation

A **graph** is associated to the system:

- each node corresponds with an inertial elment;
- each edge corresponds to a spring;

D is the coincidence matrix and describes the graph topology

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{p}\\ \boldsymbol{\varepsilon}\end{pmatrix} = \begin{bmatrix}0 & -\mathbf{D}^\top\\ \mathbf{D} & 0\end{bmatrix}\begin{pmatrix}\partial_{\mathbf{p}}H\\ \partial_{\boldsymbol{\varepsilon}}H\end{pmatrix}.$$

- $\boldsymbol{\varepsilon} = (\varepsilon_1 \quad \varepsilon_2)^\top$ spring elongations;
- $H = \frac{1}{2}k||\boldsymbol{\varepsilon}||^2 + \frac{1}{2m}||\mathbf{p}||^2$.

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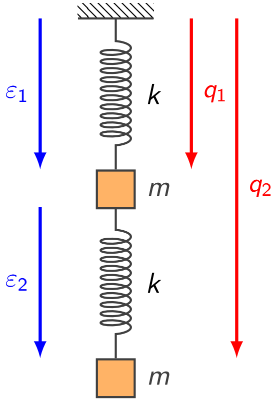

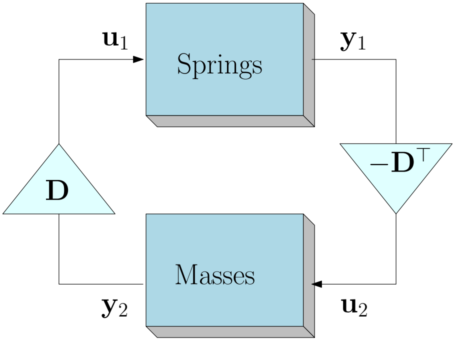

This formulation corresponds to a mixed finite element discretization.

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The port-Hamiltonian formulation for longitudinal waves[2]

The energy doesn't depend on q but only on its derivative $\varepsilon = \partial_x q$:

$$H(p, \varepsilon) = \frac{1}{2} \int_0^L \frac{p^2}{\rho} + k\varepsilon^2 \mathrm{d}x.$$

What if we write the equations using the variables that explicitly appear in the energy?

[2] van der Schaft and B. Maschke, "Hamiltonian formulation of distributed-parameter systems with boundary energy flow".

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Port-Hamiltonian formulation

Two coupled conservation laws are obtained

$$\frac{\partial}{\partial t} \begin{pmatrix} \varepsilon \\ p \end{pmatrix} = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix} \begin{pmatrix} \delta_\varepsilon H \\ \delta_p H \end{pmatrix}, \qquad \begin{pmatrix} \delta_\varepsilon H \\ \delta_p H \end{pmatrix} := \begin{bmatrix} k & 0 \\ 0 & \rho^{-1} \end{bmatrix} \begin{pmatrix} \varepsilon \\ p \end{pmatrix},$$

Stress σ,
Velocity v.

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Port-Hamiltonian formulation

The system can be written using velocity and stress only

$$\begin{bmatrix} c & 0 \\ 0 & \rho \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix} \begin{pmatrix} \sigma \\ v \end{pmatrix}, \qquad c := \frac{1}{k} \text{ is the compliance.}$$

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Power balance across the boundary and causalities

Power balance: $\dot{H} = v(L)\sigma(L) - v(0)\sigma(0) = v\ \sigma \cdot n\big|_{\partial[0,L]}$.

Possible causalities

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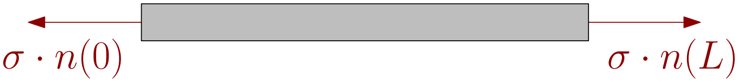
Possible causalities

Free-free (Neumann):

- Input given by the Neumann condition $\mathbf{u}_N = \sigma \cdot n|_{\partial[0,L]}$
- Output given by the Dirichlet condition $\mathbf{y}_D = v|_{\partial[0,L]}$



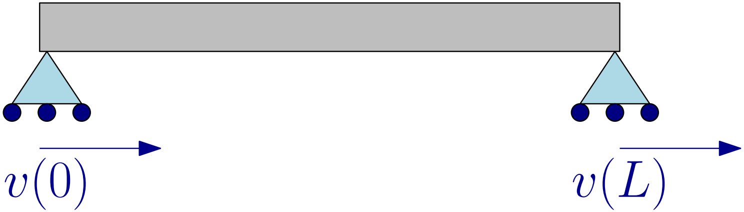
Power balance across the boundary and causalities

Power balance: $\dot{H} = v(L)\sigma(L) - v(0)\sigma(0) = v\ \sigma \cdot n\big|_{\partial[0,L]}$.

Possible causalities

Clamped-clamped (Dirichlet):

- Input given by the Dirichlet condition $\mathbf{u}_D = v|_{\partial[0,L]}$
- Output given by the Neumann condition $\mathbf{y}_N = \sigma \cdot n|_{\partial[0,L]}$



Power balance across the boundary and causalities

Power balance: $\dot{H} = v(L)\sigma(L) - v(0)\sigma(0) = v\ \sigma \cdot n\big|_{\partial[0,L]}.$

Possible causalities

Cantilever (mixed) $\partial[0, L] = \Gamma_D \cup \Gamma_N$

$$u_N = \sigma \cdot n|_{\Gamma_N}, \qquad y_D = v|_{\Gamma_N},$$
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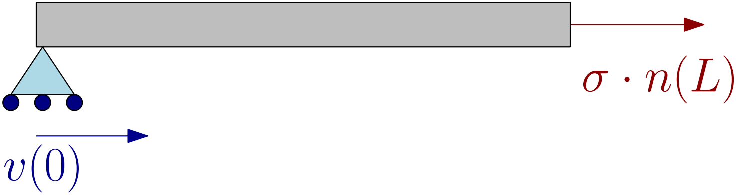

Discretization via mixed finite elements

The discretization proceeds in three steps[3]:

- take the weak formulation;
- perform integration by parts (depending on the causality);
- project on a finite element basis.

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Discretization via mixed finite elements

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- take the weak formulation;
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- project on a finite element basis.

Let's introduce the L^2 inner product ($\Omega = [0, L]$):

$$(f,\, g)_\Omega = \int_0^L f \cdot g \,\mathrm{d}x, \qquad (f,\, g)_{\partial\Omega} = f\; g \cdot n\big|_0^L.$$

and the weak formulation:

$$\begin{aligned}(\xi_v,\, \rho\,\partial_t v)_\Omega &= (\xi_v,\, \partial_x \sigma)_\Omega,\\ (\xi_\sigma,\, c\,\partial_t \sigma)_\Omega &= (\xi_\sigma,\, \partial_x v)_\Omega.\end{aligned}$$

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The primal dual structure

First weak formulation: Neumann natural control

Find $\sigma \in L^2(\Omega),\ v \in H^1(\Omega)$

$$\begin{aligned}
(\xi_\sigma,\, c\,\partial_t \sigma)_\Omega &= +(\xi_\sigma,\, \partial_x v)_\Omega, & \forall\, \xi_\sigma \in L^2(\Omega),\\
(\xi_v,\, \rho\,\partial_t v)_\Omega &= -(\partial_x \xi_v,\, \sigma)_\Omega + (\xi_v,\, \mathbf{u}_N)_{\partial\Omega}, & \forall\, \xi_v \in H^1(\Omega).
\end{aligned}$$

The primal dual structure

Second weak formulation: Dirichlet natural control

Find $\sigma \in H^1(\Omega),\; v \in L^2(\Omega)$

$$
\begin{aligned}
(\xi_\sigma,\, c\,\partial_t \sigma)_\Omega &= -(\partial_x \xi_\sigma,\, \sigma)_\Omega + (\xi_\sigma,\, \mathbf{u}_D)_{\partial\Omega}, && \forall\, \xi_\sigma \in H^1(\Omega),\\
(\xi_v,\, \rho\,\partial_t v)_\Omega &= +(\xi_v,\, \partial_x \sigma)_\Omega, && \forall\, \xi_v \in L^2(\Omega).
\end{aligned}
$$

Finite element basis

The basis for the two variables need to be different to avoid spurious mode

$$\sigma(x,t) = \sum_{i=1}^{N_\sigma} \varphi_\sigma^i(x)\sigma_i(t), \qquad v(x,t) = \sum_{i=1}^{N_v} \varphi_v^i(x) v_i(t)$$

The bases functions span the corresponding finite element space

$$\sigma \in \mathcal{S} = \mathrm{span}\{\varphi_\sigma^1, \ldots, \varphi_\sigma^{N_\sigma}\},$$
$$v \in \mathcal{V} = \mathrm{span}\{\varphi_v^1, \ldots, \varphi_v^{N_v}\},$$

Choice of the finite element basis (Neumann control)

In this formulation

- $v \in H^1(\Omega)$. **Lagrange elements** (just like in the static case) can be used.
- $\sigma \in L^2(\Omega)$. Which finite element space to choose?

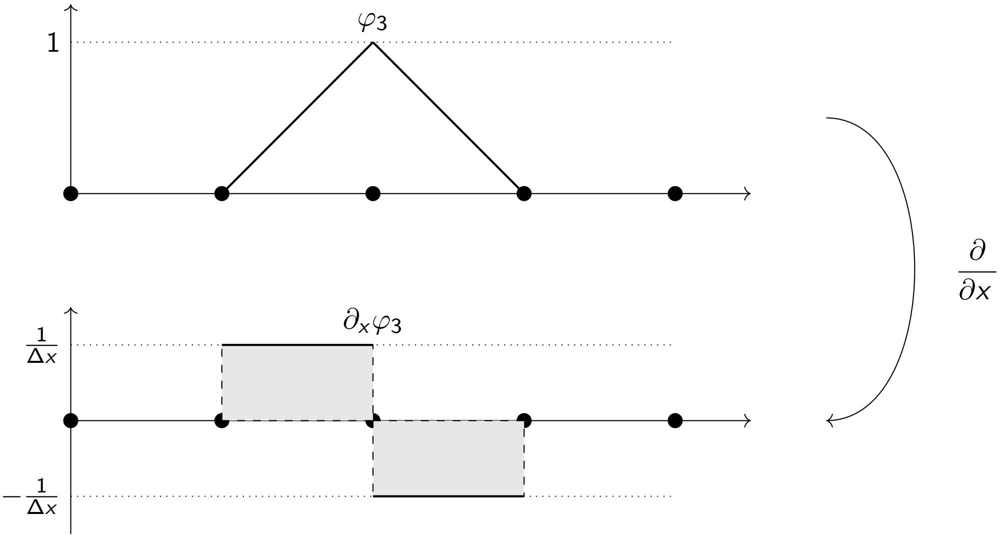
Remind the second equation reads

$$(\xi_\sigma,\, c\,\partial_t \sigma)_\Omega = (\xi_\sigma,\, \partial_x v)_\Omega.$$

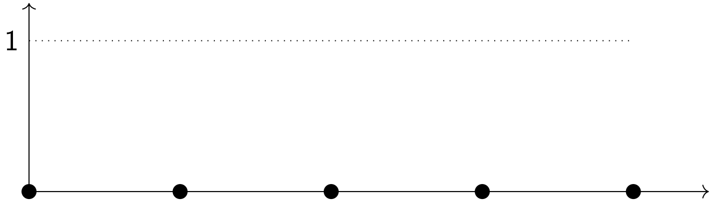
For this equation to hold pointwise, the finite element space should satisfy

$$\partial_x \mathcal{V} \subset \mathcal{S}, \implies c\,\partial_t \sigma = \partial_x v, \qquad \text{(if } c \text{ is smooth).}$$

The derivative of a Lagrange space



The Discontinous Galerkin space $\mathbb{DG}_0$



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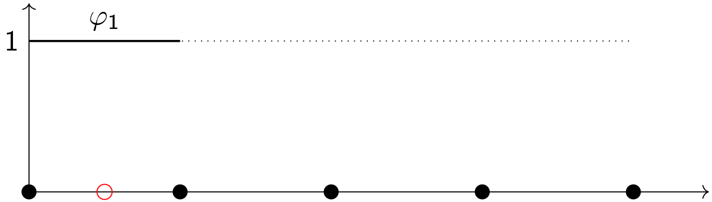

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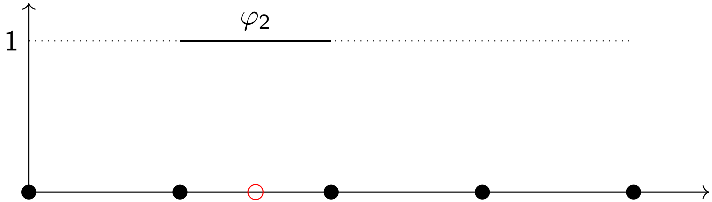

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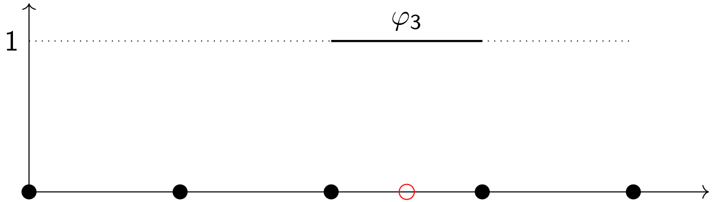

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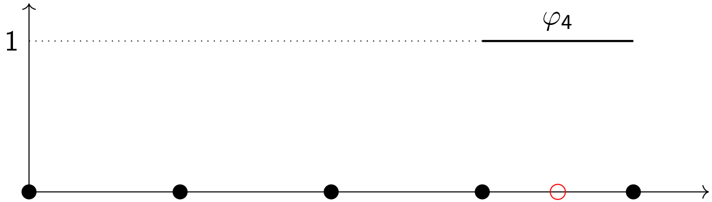

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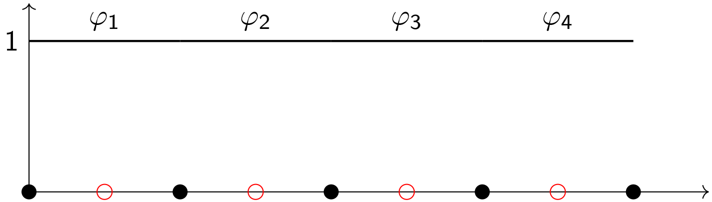

It holds $\partial_x \mathbb{L}_1 \subset \mathbb{DG}_0$. This choice guarantees stability of the formulation.

This is a particular instance of a much more general mathematical construction (subcomplex of an Hilbert complex).

Algebraic system: dynamics

Formulation with Neumann natural control

$$\begin{bmatrix} \mathbf{M}_c^\sigma & 0 \\ 0 & \mathbf{M}_\rho^v \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{D} \\ -\mathbf{D}^\top & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} + \begin{bmatrix} 0 \\ \mathbf{Tr}^\top \end{bmatrix} \mathbf{u}_N,$$

$$\mathbf{y}_D = \begin{bmatrix} 0 & \mathbf{Tr} \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix}.$$

The matrices are computed as follows

$$[\mathbf{M}_\rho^v]_{ij} = \int_0^L \rho \varphi_v^i \cdot \varphi_v^j \,\mathrm{d}x, \qquad [\mathbf{M}_c^\sigma]_{ij} = \int_0^L c \varphi_\sigma^i \cdot \varphi_\sigma^j \,\mathrm{d}x, \qquad [\mathbf{D}]_{ij} = \int_0^L \varphi_\sigma^i \cdot \frac{\partial \varphi_v^j}{\partial x} \,\mathrm{d}x.$$

Tr is a trace matrix

$$\mathbf{Tr} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

The dual formulation

For the 1D wave equation, the dual formulation is completely symmetrical.

Formulation with Dirichlet natural control

$$\begin{bmatrix} \mathbf{M}_c^\sigma & 0 \\ 0 & \mathbf{M}_\rho^v \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{D}^\top \\ \mathbf{D} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} + \begin{bmatrix} \mathbf{Tr}_n^\top \\ 0 \end{bmatrix} \mathbf{u}_D,$$

$$\mathbf{y}_N = \begin{bmatrix} \mathbf{Tr}_n & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix}.$$

$\mathbf{Tr}_n$ is the normal trace matrix

$$\mathbf{Tr}_n = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

Mixed boundary conditions

Partition of the boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ (in 1D each subpartition is 1 point).

$$u_N = \sigma \cdot n\big|_{\Gamma_N}, \qquad u_D = v\big|_{\Gamma_D}.$$

Then the resulting system is a DAE (differential algebraic equation).

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$$u_N = \sigma \cdot n|_{\Gamma_N}, \qquad u_D = v|_{\Gamma_D}.$$

Then the resulting system is a DAE (differential algebraic equation).

Primal formulation (mixed control)

$$\mathrm{Diag}\begin{bmatrix}\mathbf{M}_c^\sigma \\ \mathbf{M}_\rho^v \\ 0\end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \lambda_N\end{pmatrix} = \begin{bmatrix} 0 & \mathbf{D} & 0 \\ -\mathbf{D}^\top & 0 & \mathbf{Tr}_{\Gamma_D}^\top \\ 0 & -\mathbf{Tr}_{\Gamma_D} & 0 \end{bmatrix}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \lambda_N\end{pmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{Tr}_{\Gamma_N}^\top & 0 \\ 0 & 1\end{bmatrix}\begin{pmatrix}u_N \\ u_D\end{pmatrix},$$

$$\begin{pmatrix}y_D \\ y_N\end{pmatrix} = \begin{bmatrix} 0 & \mathbf{Tr}_{\Gamma_N} & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \lambda_N\end{pmatrix}.$$

Mixed boundary conditions

Partition of the boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ (in 1D each subpartition is 1 point).

$$u_N = \sigma \cdot n|_{\Gamma_N}, \qquad u_D = v|_{\Gamma_D}.$$

Then the resulting system is a DAE (differential algebraic equation).

Dual formulation (mixed control)

$$\mathrm{Diag}\begin{bmatrix} \mathbf{M}_c^\sigma \\ \mathbf{M}_\rho^v \\ 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{s} \\ \mathbf{v} \\ \lambda_D \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{D}^\top & \mathbf{Tr}_{n,\Gamma_N}^\top \\ \mathbf{D} & 0 & 0 \\ -\mathbf{Tr}_{n,\Gamma_N} & 0 & 0 \end{bmatrix}\begin{pmatrix} \mathbf{s} \\ \mathbf{v} \\ \lambda_D \end{pmatrix} + \begin{bmatrix} \mathbf{Tr}_{n,\Gamma_D}^\top & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} u_D \\ u_N \end{pmatrix},$$

$$\begin{pmatrix} y_N \\ y_D \end{pmatrix} = \begin{bmatrix} \mathbf{Tr}_{n,\Gamma_D} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} \mathbf{s} \\ \mathbf{v} \\ \lambda_D \end{pmatrix}.$$

Preservation of the power balance

For both the primal and the dual formulation, the energy is given by

$$H = \frac{1}{2}\mathbf{v}^\top \mathbf{M}_\rho^v \mathbf{v} + \frac{1}{2}\mathbf{s}^\top \mathbf{M}_c^\sigma \mathbf{s}.$$

For the different causalities, the time derivative gives

$$\begin{aligned}
\text{Neumann control}: \quad & \dot{H} = \mathbf{y}_D \cdot \mathbf{u}_N, \\
\text{Dirichlet control}: \quad & \dot{H} = \mathbf{y}_N \cdot \mathbf{u}_D, \\
\text{Mixed control}: \quad & \dot{H} = y_N \cdot u_D + y_D \cdot u_N.
\end{aligned}$$

Time integration and equivalence between different formulations

Since the obtained system is Hamiltonian the **same scheme detailed before can be used**.

The **primal port-Hamiltonian** formulation is **equivalent to the Lagrangian** formulation if the longitudinal **displacement** is reconstructed via the **trapezoidal rule**[4]

$$\mathbf{q}^{n+1} = \mathbf{q}^{n} + \Delta t \mathbf{v}^{n+\frac{1}{2}}.$$

[4] Brugnoli and Mehrmann, "On the discrete equivalence of Lagrangian, Hamiltonian and mixed finite element formulations for linear wave phenomena".

Summarizing

This framework is based on system theory to describe interaction with the environment:

- it formalizes the idea of interconnection by treating the boundary conditions as input/output;
- it highlights the primal-dual structure of physical systems;
- it applies to multi-physical phenomena;
- numerical schemes can take inspiration from system theory (finite elements are also based on the idea of interconnection);

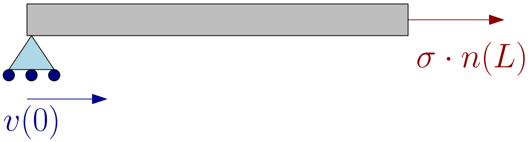
Summarizing

This framework is based on system theory to describe interaction with the environment:

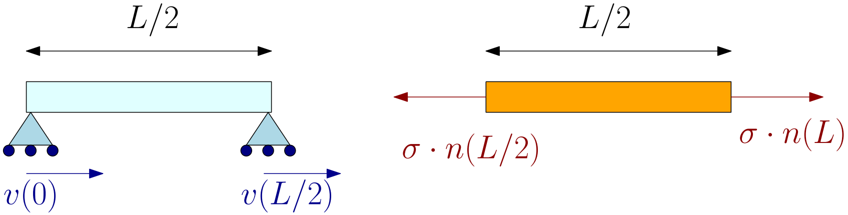
- it formalizes the idea of interconnection by treating the boundary conditions as input/output;
- it highlights the primal-dual structure of physical systems;
- it applies to multi-physical phenomena;
- numerical schemes can take inspiration from system theory (finite elements are also based on the idea of interconnection);

Mixed boundary conditions via interconnection

Consider again the cantilever bar

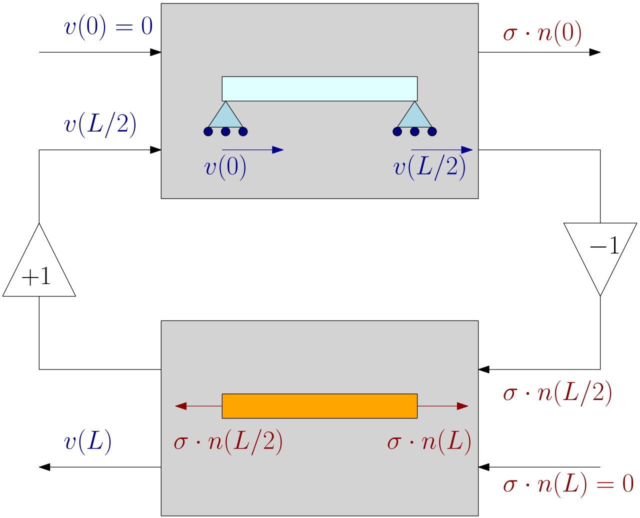


The system can be split into two parts with opposite causalities



Cantilever bar as two interconnected systems

The cantilever bar is then obtained by interconnection



Algebraic interconnection

The left part (l) is described by the dual formulation (Dirichlet bcs)

$$\begin{aligned} \mathbf{M}_{\mathrm{l}}\dot{\mathbf{x}}_{\mathrm{l}} &= \mathbf{J}_{\mathrm{l}}\mathbf{x}_{\mathrm{l}} + \mathbf{B}_{\mathrm{l}}u, \\ y &= \mathbf{B}_{\mathrm{l}}^{\top}\mathbf{x}_{\mathrm{l}}. \end{aligned}$$

The right part (r) is described by the primal formulation (Neumann bcs)

$$\begin{aligned} \mathbf{M}_{\mathrm{r}}\dot{\mathbf{x}}_{\mathrm{r}} &= \mathbf{J}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}} + \mathbf{B}_{\mathrm{r}}u, \\ y &= \mathbf{B}_{\mathrm{r}}^{\top}\mathbf{x}_{\mathrm{r}}. \end{aligned}$$

The interconnection is essentially Newton's third law

$$\begin{aligned} u &= y, && \text{The velocity is the same,} \\ u &= -y, && \text{The forces are opposite.} \end{aligned}$$

Interconnected system

The interconnected system can be written as follows

$$\begin{bmatrix} \mathbf{M}_\mathrm{l} & 0 \\ 0 & \mathbf{M}_\mathrm{r} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{x}}_l \\ \dot{\mathbf{x}}_r \end{pmatrix} = \begin{bmatrix} \mathbf{J}_\mathrm{l} & +\mathbf{B}_\mathrm{l}\mathbf{B}_\mathrm{r}^\top \\ -\mathbf{B}_\mathrm{r}\mathbf{B}_\mathrm{l}^\top & \mathbf{J}_\mathrm{r} \end{bmatrix} \begin{pmatrix} \mathbf{x}_l \\ \mathbf{x}_r \end{pmatrix}.$$

All the boundary conditions are weakly enforced.

Summary

Lagrangian and Hamiltonian form of a bar under axial loading

Port-Hamiltonian formalism

To go further: the $\mathbb{R}^d$ case

Multidimensional wave equation

Wave equation in $\Omega \subset \mathbb{R}^d$ (∂_{xx} becomes the Laplacian)

$$\begin{bmatrix} c & 0 \\ 0 & \rho \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\sigma} \\ v \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ v \end{pmatrix}.$$

Two different input causality

- Neumann control $u_N = \boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\partial\Omega}, \quad y_D = v|_{\partial\Omega}$.
- Dirichlet control $u_D = v|_{\partial\Omega}, \quad y_N = \boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\partial\Omega}$.

Input and output are now infinite dimensional:

- $v|_{\partial\Omega} \in H^{1/2}(\partial\Omega) = \mathrm{tr}(H^1(\Omega)) := \{u \in L^2(\partial\Omega) \mid \exists v \in H^1(\Omega) : \mathrm{tr}(v) = u\}$,
- $\boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ the corresponding dual space.

The following space is also needed:

$$H^{\mathrm{div}}(\Omega) = \{\boldsymbol{\sigma} \in L^2(\Omega, \mathbb{R}^d) \mid \mathrm{div}(\boldsymbol{\sigma}) \in L^2(\Omega)\}.$$

Weak formulations

In higher space dimensions, the two formulations are **not symmetrical anymore**.

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First weak formulation: Neumann control

Find $\boldsymbol{\sigma} \in L^2(\Omega, \mathbb{R}^d)$, $v \in H^1(\Omega)$ such that

$$\begin{aligned}
(\boldsymbol{\xi}_\sigma,\, c\partial_t \boldsymbol{\sigma})_\Omega &= +(\boldsymbol{\xi}_\sigma,\, \operatorname{grad} v)_\Omega, && \forall \boldsymbol{\xi}_\sigma \in L^2(\Omega, \mathbb{R}^d),\\
(\xi_v,\, \rho\partial_t v)_\Omega &= -(\operatorname{grad} \xi_v,\, \boldsymbol{\sigma})_\Omega + (\xi_v,\, u_N)_{\partial\Omega}, && \forall \xi_v \in H^1(\Omega),\\
(\xi_\partial,\, y_D)_{\partial\Omega} &= (\xi_\partial,\, v)_{\partial\Omega}, && \forall \xi_\partial \in H^{-1/2}(\partial\Omega).
\end{aligned}$$

Weak formulations

In higher space dimensions, the two formulations are **not symmetrical anymore**.

Second weak formulation: Dirichlet control

Find $\boldsymbol{\sigma} \in H^{\text{div}}(\Omega),\ v \in L^2(\Omega)$ such that

$$
\begin{aligned}
(\boldsymbol{\xi}_\sigma,\, c\partial_t \boldsymbol{\sigma})_\Omega &= -(\operatorname{div} \boldsymbol{\xi}_\sigma,\, v)_\Omega + (\boldsymbol{\xi}_\sigma \cdot \boldsymbol{n},\, u_D)_{\partial\Omega}, && \forall \boldsymbol{\xi}_\sigma \in H^{\text{div}}(\Omega, \mathbb{R}^d),\\
(\xi_p,\, \rho\partial_t v)_\Omega &= +(\xi_p,\, \operatorname{div} \boldsymbol{\sigma})_\Omega, && \forall \xi_p \in L^2(\Omega),\\
(\xi_\partial,\, y_N)_{\partial\Omega} &= (\xi_\partial,\, \boldsymbol{\sigma} \cdot \boldsymbol{n})_{\partial\Omega}, && \forall \xi_\partial \in H^{1/2}(\partial\Omega).
\end{aligned}
$$

Finite element spaces

For the coenergy variable $\boldsymbol{\sigma}$, we need to use a vector valued space

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \sum_{i=1}^{N_\sigma} \boldsymbol{\varphi}_\sigma^i(\boldsymbol{x}) e_\sigma^i(t), \qquad v(\boldsymbol{x}, t) = \sum_{i=1}^{N_v} \varphi_v^i(\boldsymbol{x}) v^i(t).$$

The input and output are discretized using the same basis.

Neumann control

$$u_N(\boldsymbol{s}, t) = \sum_{i=1}^{N_\partial} \varphi_\partial^i(\boldsymbol{x}) u_D^i(t), \qquad y_D(\boldsymbol{s}, t) = \sum_{i=1}^{N_\partial} \varphi_\partial^i(\boldsymbol{x}) y_N^i(t),$$

where $\boldsymbol{s}$ designates a coordinate parametrization of the boundary.

$$u_N, \ y_D \in \xi_\partial = \mathrm{span}\{\varphi_\partial^1, \dots, \varphi_\partial^{N_\partial}\}.$$

This guarantees that collocated input and output matrices are obtained.

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Dirichlet control

$$u_D(\boldsymbol{s}, t) = \sum_{i=1}^{N_\partial} \varphi_\partial^i(\boldsymbol{s}) u_D^i(t), \qquad y_N(\boldsymbol{s}, t) = \sum_{i=1}^{N_\partial} \varphi_\partial^i(\boldsymbol{s}) y_N^i(t),$$

where $\boldsymbol{s}$ designates a coordinate parametrization of the boundary.

$$u_D, \; y_N \in \xi_\partial = \mathrm{span}\{\varphi_\partial^1, \dots, \varphi_\partial^{N_\partial}\}.$$

This guarantees that collocated input and output matrices are obtained.

Choice of the finite element basis (Neumann control)

Neumann control: $(\boldsymbol{\xi}_\sigma,\, c\,\partial_t \boldsymbol{\sigma})_\Omega = (\boldsymbol{\xi}_\sigma,\, \operatorname{grad} v)_\Omega, \qquad \operatorname{grad} \mathcal{V} \subset \mathcal{S}.$

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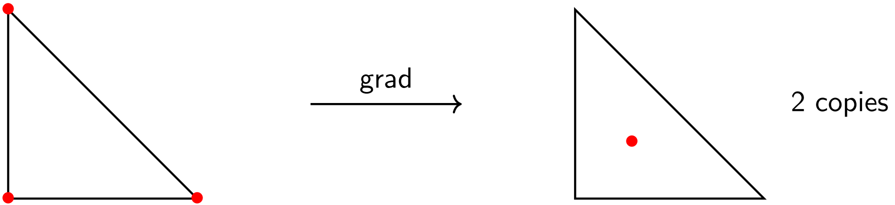


$\mathbb{L}_1$-element:

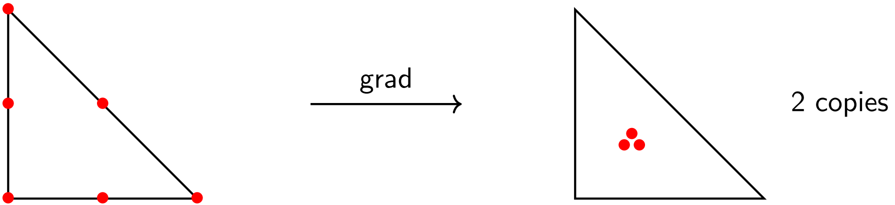
- K = triangle,
- $P_K := \{a_0 + a_1 x + a_2 y\}$,
- $\Sigma_K := \{\text{evaluation on vertices}\}$.

$\mathbb{DG}_0$-element:

- K = triangle,
- $P_K := \{a_0\}$,
- $\Sigma_K := \{\text{evaluation on centroid}\}$.

Choice of the finite element basis (Neumann control)

Neumann control: $(\boldsymbol{\xi}_\sigma, \, c\,\partial_t \boldsymbol{\sigma})_\Omega = (\boldsymbol{\xi}_\sigma, \, \operatorname{grad} v)_\Omega$, $\qquad \operatorname{grad} \mathcal{V} \subset \mathcal{S}$.



$\mathbb{L}_2$-element:

- K = triangle,
- $P_K := \{\cdots + a_3 x^2 + a_4 xy + a_5 y^2\}$,
- $\Sigma_K := \{$evaluation on vertices and midpoints$\}$.

$\mathbb{DG}_1$-element:

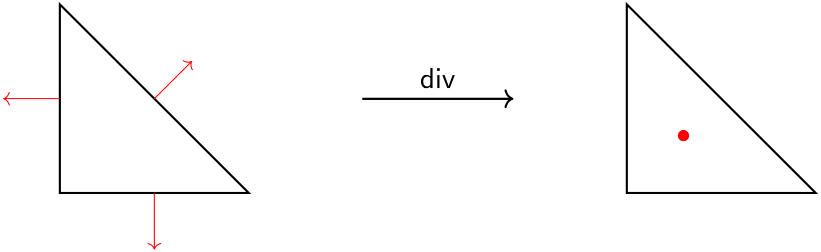
- K = triangle,
- $P_K := \{a_0 + a_1 x + a_2 y\}$,
- $\Sigma_K := \{$evaluation on 3 nodes$\}$.

Choice of the finite element basis (Dirichlet control)

Dirichlet control: $(\xi_p,\, \rho\,\partial_t v)_\Omega = (\xi_p,\, \operatorname{div}\boldsymbol{\sigma})_\Omega, \qquad \operatorname{div}\mathcal{S} \subset \mathcal{V}.$

Choice of the finite element basis (Dirichlet control)

Dirichlet control: $(\xi_p, \rho\,\partial_t v)_\Omega = (\xi_p, \operatorname{div}\boldsymbol{\sigma})_\Omega, \qquad \operatorname{div}\mathcal{S} \subset \mathcal{V}.$



$\mathbb{RT}_0$ (Raviart Thomas)-element:

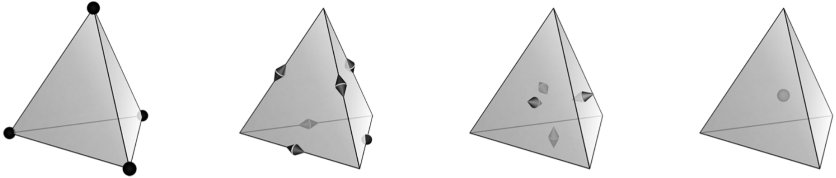
- K = triangle,
- $P_K := \left\{ \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + a_2 \begin{pmatrix} x \\ y \end{pmatrix} \right\}$,
- $\Sigma_K :=$ {integrals over faces}.

$\mathbb{DG}_0$-element:

- K = triangle,
- $P_K := \{a_0\}$,
- $\Sigma_K :=$ {evaluation on centroid}.

Finite element exterior calculus

To obtain stable formulations, finite element exterior calculus can be used[5].



The Whitney forms (1957).

- connection with differential geometry (coordinate free treatment);
- unifying framework for physics;
- clear separation of topological and metrical operations.

[5] Brugnoli, Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus".

Algebraic systems

Formulation with Neumann control

$$\begin{bmatrix} \mathbf{M}_c^\sigma & 0 \\ 0 & \mathbf{M}_\rho^v \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{D}_{\mathrm{grad}} \\ -\mathbf{D}_{\mathrm{grad}}^\top & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} + \begin{bmatrix} 0 \\ \mathbf{B}_v \end{bmatrix} \mathbf{u}_N,$$

$$\mathbf{M}_\partial \mathbf{y}_D = \begin{bmatrix} 0 & \mathbf{B}_v^\top \end{bmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{s} \end{pmatrix}.$$

The matrices are computed as follows

$$[\mathbf{D}_{\mathrm{grad}}]_{ij} = \int_\Omega \varphi_\sigma^i \cdot \mathrm{grad}\, \varphi_v^j \, \mathrm{d}\Omega, \qquad [\mathbf{B}_v]_{ij} = \int_{\partial\Omega} \varphi_v^i \varphi_\partial^j \, \mathrm{d}\Gamma.$$

Matrix $\mathbf{B}_v$ can be decomposed using the trace matrix $\mathbf{B}_v = \mathbf{Tr}^\top \mathbf{\Psi}_v$.

Algebraic systems

Formulation with Dirichlet control

$$
\begin{bmatrix} \mathbf{M}_c^\sigma & 0 \\ 0 & \mathbf{M}_\rho^v \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{D}_{\mathrm{div}}^\top \\ \mathbf{D}_{\mathrm{div}} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} + \begin{bmatrix} \mathbf{B}_\sigma \\ 0 \end{bmatrix} \mathbf{u}_D,
$$

$$
\mathbf{M}_\partial \mathbf{y}_N = \begin{bmatrix} \mathbf{B}_\sigma^\top & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix}.
$$

The matrices are computed as follows

$$
[\mathbf{D}_{\mathrm{div}}]_{ij} = \int_\Omega \varphi_v^i \, \mathrm{div}\, \boldsymbol{\varphi}_\sigma^j \, \mathrm{d}\Omega, \qquad [\mathbf{B}_\sigma]_{ij} = \int_{\partial\Omega} (\boldsymbol{\varphi}_\sigma^i \cdot \boldsymbol{n}) \, \varphi_\partial^j \, \mathrm{d}\Gamma.
$$

Matrix $\mathbf{B}_\sigma$ can be decomposed using the trace matrix $\mathbf{B}_\sigma = \mathbf{Tr}^\top \boldsymbol{\Psi}_\sigma$.

Mixed boundary control

grad formulation (mixed control)

$$\mathrm{Diag}\begin{bmatrix}\mathbf{M}_\rho^v \\ \mathbf{M}_c^\sigma \\ 0\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \boldsymbol{\lambda}_N\end{pmatrix} = \begin{bmatrix}0 & \mathbf{D}_{\mathrm{grad}} & 0 \\ -\mathbf{D}_{\mathrm{grad}}^\top & 0 & \mathbf{B}_{v,\Gamma_D} \\ 0 & -\mathbf{B}_{v,\Gamma_D}^\top & 0\end{bmatrix}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \boldsymbol{\lambda}_N\end{pmatrix} + \begin{bmatrix}0 & 0 \\ \mathbf{B}_{v,\Gamma_N} & 0 \\ 0 & \boldsymbol{M}_{\Gamma_D}\end{bmatrix}\begin{pmatrix}\boldsymbol{u}_N \\ \boldsymbol{u}_D\end{pmatrix},$$

$$\begin{bmatrix}\boldsymbol{M}_{\Gamma_N} & 0 \\ 0 & \boldsymbol{M}_{\Gamma_D}\end{bmatrix}\begin{pmatrix}\mathbf{y}_D \\ \mathbf{y}_N\end{pmatrix} = \begin{bmatrix}0 & \mathbf{B}_{v,\Gamma_N}^\top & 0 \\ 0 & 0 & \boldsymbol{M}_{\Gamma_D}\end{bmatrix}\begin{pmatrix}\mathbf{v} \\ \mathbf{s} \\ \boldsymbol{\lambda}_N\end{pmatrix}.$$

For solvability, matrix $\mathbf{B}_{v,\Gamma_D}$ should satisfy an inf-sup condition.

$$\inf_{\boldsymbol{\lambda}_N \in \mathbb{R}^{N_\partial}} \sup_{\mathbf{v}\in\mathbb{R}^{N_v}} \frac{\mathbf{v}^\top \mathbf{B}_{v,\Gamma_D}\boldsymbol{\lambda}_N}{||\mathbf{v}||_2 ||\boldsymbol{\lambda}_N||_2} \geq \beta_v > 0, \qquad \mathbf{v} \neq 0, \quad \boldsymbol{\lambda}_N \neq 0.$$

The coefficient β_v is the smallest singular value of $\mathbf{B}_{v,\Gamma_D}$.

Mixed boundary control

div formulation

$$\mathrm{Diag}\begin{bmatrix}\mathbf{M}_c^{\sigma}\\ \mathbf{M}_\rho^{v}\\ 0\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{s}\\ \mathbf{v}\\ \boldsymbol{\lambda}_D\end{pmatrix} = \begin{bmatrix}0 & -\mathbf{D}_{\mathrm{div}}^\top & \mathbf{B}_{\sigma,\Gamma_N}\\ \mathbf{D}_{\mathrm{div}} & 0 & 0\\ -\mathbf{B}_{\sigma,\Gamma_N}^\top & 0 & 0\end{bmatrix}\begin{pmatrix}\mathbf{s}\\ \mathbf{v}\\ \boldsymbol{\lambda}_D\end{pmatrix} + \begin{bmatrix}\mathbf{B}_{\sigma,\Gamma_D} & 0\\ 0 & 0\\ 0 & \boldsymbol{M}_{\Gamma_N}\end{bmatrix}\begin{pmatrix}\boldsymbol{u}_D\\ \boldsymbol{u}_N\end{pmatrix},$$

$$\begin{bmatrix}\boldsymbol{M}_{\Gamma_D} & 0\\ 0 & \boldsymbol{M}_{\Gamma_N}\end{bmatrix}\begin{pmatrix}\mathbf{y}_N\\ \mathbf{y}_D\end{pmatrix} = \begin{bmatrix}\mathbf{B}_{\sigma,\Gamma_D}^\top & 0 & 0\\ 0 & 0 & \boldsymbol{M}_{\Gamma_N}\end{bmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{s}\\ \boldsymbol{\lambda}_D\end{pmatrix}.$$

For solvability, matrix $\mathbf{B}_{\sigma,\Gamma_N}$ should satisfy an inf-sup condition.

$$\inf_{\boldsymbol{\lambda}_D\in\mathbb{R}^{N_\partial}}\ \sup_{\mathbf{v}\in\mathbb{R}^{N_\sigma}}\frac{\mathbf{v}^\top\mathbf{B}_{\sigma,\Gamma_N}\boldsymbol{\lambda}_D}{||\mathbf{v}||_2||\boldsymbol{\lambda}_D||_2}\geq\beta_\sigma>0,\qquad \mathbf{v}\neq 0,\quad \boldsymbol{\lambda}_N\neq 0.$$

The coefficient β_σ is the smallest singular value of $\mathbf{B}_{\sigma,\Gamma_N}$.

Power Flow

The discrete systems satisfy:

$$
\begin{aligned}
\text{Neumann control}: \quad & \dot{H} = \mathbf{y}_D^\top \mathbf{M}_\partial \mathbf{u}_N, \\
\text{Dirichlet control}: \quad & \dot{H} = \mathbf{y}_N^\top \mathbf{M}_\partial \mathbf{u}_D, \\
\text{Mixed control}: \quad & \dot{H} = \mathbf{y}_N^\top \mathbf{M}_{\Gamma_D} \mathbf{u}_D + \mathbf{y}_D^\top \mathbf{M}_{\Gamma_N} \mathbf{u}_N.
\end{aligned}
$$

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