Dual field mixed and hybrid FE for port-Hamiltonian systems

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Outline

Introduction

Dirac structures

Port-Hamiltonian systems
- The de Rham complex case: wave and Maxwell equations
- Non linear elastodynamics
- Linear elasticity and plate models

Dual field discretization of Maxwell equations
- Galerkin mixed discretization
- Hybrid scheme

Outline

Port-Hamiltonian systems and physics

Hamiltonian systems can be used to describe a wide variety of problems

- Solid mechanics
- Fluid mechanics
- Electromagnetism
- Thermodynamics (metriplectic systems, irreversible Hamiltonian systems)

Port-Hamiltonian systems (in their simplest form) are described by[1]:

- A **Dirac manifold**[2] (embeds the **topology** of the network).
- An **Hamiltonian** (incorporates the **metric** information).

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[1] Schaft and Maschke, "Hamiltonian formulation of distributed-parameter systems with boundary energy flow".

[2] Courant, "Dirac manifolds".

Outline

Dirac Structures

Definition (Dirac structure)

Given a vector space E and its dual $F = E'$ with duality product $\langle \cdot \,|\cdot \rangle_{F,E} : F \times E \to \mathbb{R}$, consider the symmetric bilinear form on $B := F \times E$:

$$\langle\langle\, (f^1, e^1), (f^2, e^2)\, \rangle\rangle := \langle f^2 \,|e^1\rangle_{F,E} + \langle f^1 \,|e^2\rangle_{F,E}, \qquad (f^i, e^i) \in B,\ i = 1, 2.$$

A Dirac structure is a subspace $\mathcal{D} \subset B$ such that $\mathcal{D} = \mathcal{D}^{[\perp]}$ w.r.t. $\langle\langle \cdot, \cdot \rangle\rangle$.

Property: Dirac structures are composable: $D_{\text{int}} = D_1 \circ D_2$

Examples

Finite dimensional case

Let $\mathbf{J} = -\mathbf{J}^\top \in \mathbb{R}^{n\times n}$. The following is a Dirac structure over $B = \mathbb{R}^n \times \mathbb{R}^n$

$$D := \mathrm{Graph}(\mathbf{J}) := \{(\mathbf{f}, \mathbf{e}) \in B \quad | \quad \mathbf{f} = \mathbf{J}\mathbf{e} \quad \forall \mathbf{e} \in \mathbb{R}^n\}.$$

Infinite dimensional case

Let E be a real Hilbert space and $F = E'$ its dual. Let $\mathcal{J} = \mathcal{L}(E, F)$ be a skew-dual bounded operator

$$\langle \mathcal{J} e^1 \,|e^2\rangle_{F,E} = -\langle \mathcal{J} e^2 \,|e^1\rangle_{F,E} \qquad \forall e^1, e^2 \in E,$$

then the following is a Dirac structure over $B = F \times E$

$$D := \mathrm{Graph}(\mathcal{J}) := \{(f, e) \in B \quad | \quad f = \mathcal{J} e \quad \forall e \in E\}.$$

Geometric Dirac structure: duality pairing of forms

Let $\Lambda^k(\Omega)$ is the space of smooth k-forms on a Riemannian oriented manifold Ω

Definition (Duality product)

Given a manifold M, $\dim(M) = m$ (e.g. $M = \Omega$ or $M = \partial\Omega$) via

$$\langle \alpha^k \,|\beta^{m-k}\rangle_\Omega := \int_M \alpha^k \wedge \beta^{m-k}, \quad \alpha^k \in \Lambda^k(M), \quad \beta^{m-k} \in \Lambda^{m-k}(M), \quad k = 0, \dots, m.$$

Integration by parts formula

By the Leibniz rule and the Stokes theorem it holds

$$\langle \, \mathrm{d}\mu \,|\lambda\rangle_\Omega + (-1)^k \langle \mu \,|\, \mathrm{d}\lambda\rangle_\Omega = \langle \mathrm{tr}\,\mu \,|\, \mathrm{tr}\,\lambda\rangle_{\partial\Omega}, \qquad \mu \in \Lambda^k(\Omega), \quad \lambda \in \Lambda^{n-k-1}(\Omega).$$

Geometric Dirac structure[3]

Dirac structure for differential forms

Given $p+q=n+1$ consider

- the flows $(f_1^p,\, f_2^q,\, f_\partial^{n-p}) \in F = \Lambda^p(\Omega) \times \Lambda^q(\Omega) \times \Lambda^{n-p}(\partial\Omega)$,
- the efforts $(e_1^{n-p},\, e_2^{n-q},\, e_\partial^{n-q}) \in E = \Lambda^{n-p}(\Omega) \times \Lambda^{n-q}(\Omega) \times \Lambda^{n-q}(\partial\Omega)$,

The following subset defines a Dirac structure

$$\begin{pmatrix} f_1^p \\ f_2^q \end{pmatrix} = \begin{bmatrix} 0 & (-1)^{pq+1}\,\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{pmatrix} e_1^{n-p} \\ e_2^{n-q} \end{pmatrix}, \qquad \begin{pmatrix} f_\partial^{n-p} \\ e_\partial^{n-q} \end{pmatrix} = \begin{bmatrix} \mathrm{tr} & 0 \\ 0 & (-1)^p\,\mathrm{tr} \end{bmatrix} \begin{pmatrix} e_1^{n-p} \\ e_2^{n-q} \end{pmatrix}.$$

Proof: Use the integration by parts formula and Stokes theorem

$$\langle e_1^{n-p} \,|\, f_1^p \rangle_\Omega + \langle e_2^{n-q} \,|\, f_2^q \rangle_\Omega + \langle e_\partial^{n-q} \,|\, f_\partial^{n-p} \rangle_{\partial\Omega} = 0.$$

[3] Schaft and Maschke, "Hamiltonian formulation of distributed-parameter systems with boundary energy flow".

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Dual field discretization of Maxwell equations

Port-Hamiltonian system

Definition (Lossless port-Hamiltonian (pH) system)

A lossless port-Hamiltonian (pH) system is a tuple (D, H) where
$D \subset X_{\mathcal{S}} \times X_{\mathcal{P}} \times X'_{\mathcal{S}} \times X'_{\mathcal{P}}$ is a **Dirac structure** and $H : U \subset X_{\mathcal{S}} \to \mathbb{R}$ a **Hamiltonian**.

The trajectories $(x(t), f_P(t)) \in X_{\mathcal{S}} \times X_{\mathcal{P}}$ and $(\delta_x H, e_P(t)) \in X'_{\mathcal{S}} \times X'_{\mathcal{P}}$ satisfy

$$((\frac{d}{dt}x(t),\ f_{\mathcal{P}}),\ (\delta_x H,\ e_{\mathcal{P}})) \in D.$$

Finite dimensional case

Consider the Hamiltonian system $\mathbf{J} = -\mathbf{J}^\top$ with collocated control/observation:

$$\dot{\mathbf{x}} = \mathbf{J}\nabla H + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{B}^\top \nabla H.$$

Port-Hamiltonian system

Definition (Lossless port-Hamiltonian (pH) system)

A lossless port-Hamiltonian (pH) system is a tuple (D, H) where $D \subset X_{\mathcal{S}} \times X_{\mathcal{P}} \times X'_{\mathcal{S}} \times X'_{\mathcal{P}}$ is a **Dirac structure** and $H : U \subset X_{\mathcal{S}} \to \mathbb{R}$ a **Hamiltonian**.

The trajectories $(x(t), f_P(t)) \in X_{\mathcal{S}} \times X_{\mathcal{P}}$ and $(\delta_x H, e_P(t)) \in X'_{\mathcal{S}} \times X'_{\mathcal{P}}$ satisfy

$$((\frac{d}{dt}x(t),\ f_{\mathcal{P}}),\ (\delta_x H,\ e_{\mathcal{P}})) \in D.$$

Finite dimensional case

From the identification $\mathbf{e}_{\mathcal{P}} = \mathbf{u},\ \mathbf{f}_{\mathcal{P}} = -\mathbf{y}$ it follows

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \mathbf{f}_{\mathcal{P}} \end{pmatrix} = \begin{bmatrix} \mathbf{J} & \mathbf{B} \\ -\mathbf{B}^\top & 0 \end{bmatrix} \begin{pmatrix} \nabla H \\ \mathbf{e}_{\mathcal{P}} \end{pmatrix}$$

This means $((\dot{\mathbf{x}},\ \mathbf{f}_{\mathcal{P}}), (\nabla H,\ \mathbf{e}_{\mathcal{P}})) \in D$.

Hyperbolic boundary control systems

The following defines a Dirac structure on the oriented Riemannian manifold Ω

$$\begin{pmatrix} \partial_t \alpha^p \\ \partial_t \beta^q \end{pmatrix} = \begin{bmatrix} 0 & (-1)^{pq+1}\,\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{pmatrix} \delta_\alpha H^{n-p} \\ \delta_\beta H^{n-q} \end{pmatrix}, \qquad (-1)^p \operatorname{tr} \delta_\beta H^{n-q} = u^{n-q},$$

$$y^{n-p} = \operatorname{tr} \delta_\alpha H^{n-p},$$

where the variational derivative is defined by

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} H(\mu^k + \varepsilon \delta\mu^k) = \langle \delta_\mu H^{n-p} \,|\delta\mu^k \rangle_\Omega.$$

- the **flow** space $F = \Lambda^p(\Omega) \times \Lambda^q(\Omega) \times \Lambda^{n-p}(\partial\Omega)$

$$(f_1^p,\, f_2^q,\, f_\partial^{n-p}) := (\partial_t \alpha^p,\, \partial_t \beta^q,\, -y^{n-p})$$

- the **effort** space $E = \Lambda^{n-p}(\Omega) \times \Lambda^{n-q}(\Omega) \times \Lambda^{n-q}(\partial\Omega)$.

$$(e_1^{n-p},\, e_2^{n-q},\, e_\partial^{n-q}) := (\delta_\alpha H^{n-p},\, \delta_\beta H^{n-q},\, u^{n-q}).$$

The geometric wave and Maxwell equations $\Omega \subset \mathbb{R}^3$

Wave equation $p = 3,\ q = 1$

Hamiltonian $H = \frac{1}{2}\{(\psi^3,\, \psi^3)_\Omega + (\boldsymbol{v}^1,\, \boldsymbol{v}^1)_\Omega\}$

$$\begin{pmatrix} \partial_t \psi^3 \\ \partial_t \boldsymbol{v}^1 \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix} \begin{pmatrix} \star\psi^3 \\ \star\boldsymbol{v}^1 \end{pmatrix}, \qquad \mathrm{tr}\star\boldsymbol{v}^1 = u^2,$$

$$y^0 = \mathrm{tr}\star\psi^3.$$

The geometric wave and Maxwell equations $\Omega \subset \mathbb{R}^3$

Maxwell equations $p = 2,\ q = 2$

Hamiltonian $H = \frac{1}{2}\{(\boldsymbol{E}^2,\ \boldsymbol{E}^2)_\Omega + (\boldsymbol{H}^2,\ \boldsymbol{H}^2)_\Omega\}$

$$\begin{pmatrix} \partial_t \boldsymbol{E}^2 \\ \partial_t \boldsymbol{H}^2 \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{curl} \\ -\,\mathrm{curl} & 0 \end{bmatrix} \begin{pmatrix} \star_{\varepsilon^{-1}} \boldsymbol{E}^2 \\ \star_{\mu^{-1}} \boldsymbol{H}^2 \end{pmatrix}, \qquad \mathrm{tr} \star \boldsymbol{E}^2 = u^1,$$

$$y^1 = \mathrm{tr} \star \boldsymbol{H}^2.$$

Outline

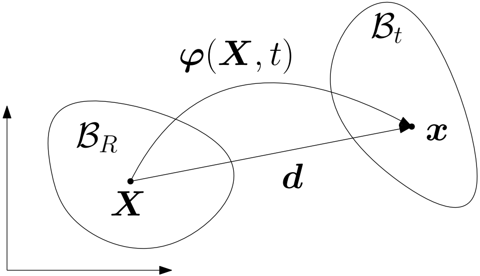
Non linear elastodynamics (material description)

Motion described by: $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$.

- Material velocity $\boldsymbol{v}(t) = \frac{\partial \varphi}{\partial t} \in \mathbb{R}^d$;
- Deformation gradient $\boldsymbol{F}(t) = \frac{\partial \varphi}{\partial \boldsymbol{X}} \in \mathbb{R}^{d \times d}$;

Energy $H = T + U$ (ρ_R is constant):

- Kinetic energy $T = \frac{1}{2} \int_{\mathcal{B}_R} \rho_R \boldsymbol{v} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{X}$;
- Deformation energy $U = \int_{\mathcal{B}_R} \rho_R \Psi(\boldsymbol{F}) \, \mathrm{d}\boldsymbol{X}$.



R: reference configuration

Euler Lagrange equations for non linear elasticity

$$\rho_R \frac{\partial \boldsymbol{v}}{\partial t} = \mathrm{div}_{\boldsymbol{X}} \boldsymbol{S}, \qquad \boldsymbol{S} := \frac{\delta U}{\delta \boldsymbol{F}} = \rho_R \frac{\partial \Psi}{\partial \boldsymbol{F}}$$ First Piola-Kirchhoff stress tensor.

Port-Hamiltonian formulation of non linear elastodynamics

The dynamics of $\boldsymbol{p} = \rho_R \boldsymbol{v}$ and $\boldsymbol{F}$ is given by the port-Hamiltonian PDE

$$\frac{\partial}{\partial t}\begin{pmatrix}\boldsymbol{p} \\ \boldsymbol{F}\end{pmatrix} = \begin{bmatrix} 0 & \mathrm{div}_{\boldsymbol{X}} \\ \nabla_{\boldsymbol{X}} & 0 \end{bmatrix}\begin{pmatrix}\delta_{\boldsymbol{p}} H \\ \delta_{\boldsymbol{F}} H\end{pmatrix}, \qquad \delta_{\boldsymbol{F}} H|_{\partial\mathcal{B}_R} = \boldsymbol{S}\cdot\boldsymbol{N}|_{\partial\mathcal{B}_R} = \boldsymbol{u},$$

$$\boldsymbol{y} = \delta_{\boldsymbol{p}} H|_{\partial\mathcal{B}_R} = \boldsymbol{v}|_{\partial\mathcal{B}_R}.$$

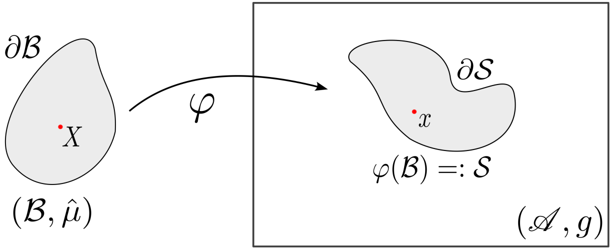
Hyperelastic constitutive law:

$$\rho_R \Psi(\boldsymbol{F}) = \frac{1}{2}(\boldsymbol{E}, \mathcal{K}\boldsymbol{E})_{\mathrm{F}}, \qquad \boldsymbol{E}(\boldsymbol{F}) = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I}) \quad \text{Green Lagrange strain}$$

$(\boldsymbol{A}, \boldsymbol{B})_{\mathrm{F}} := \mathrm{Tr}(\boldsymbol{A}^\top \boldsymbol{B})$ Frobenius inner product, $\mathcal{K}$ stiffness tensor

Geometric non linear elasticity[4]

An elastic motion is a curve in the space of smooth embeddings $\varphi_t \in \mathrm{Emb}^\infty(\mathcal{B}, \mathscr{A})$

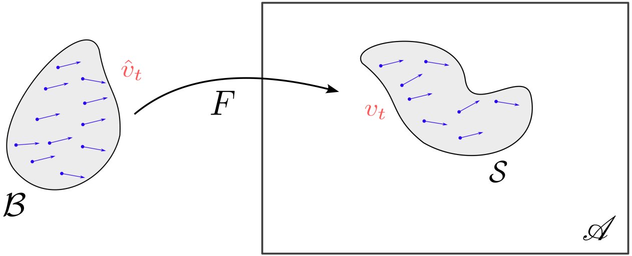


Configuration space

[4] R. Rashad et al. "Intrinsic nonlinear elasticity: An exterior calculus formulation". In: *arXiv preprint arXiv:2303.06082* (2023).

Geometric non linear elasticity[4]

Velocities are sections of tangent bundles.



Spatial and convective velocity

$$\begin{tikzcd} T\mathcal{B} \arrow[r, "F_t"] & T\mathcal{S} \\ \mathcal{B} \arrow[u, "\hat{v}_t", "\text{convective}"'] \arrow[ur, "\text{material}", "\tilde{v}_t"'] \arrow[r, "\varphi_t"'] & \mathcal{S} \arrow[u, "\text{spatial}", "v_t"'] \end{tikzcd}$$

[4] R. Rashad et al. "Intrinsic nonlinear elasticity: An exterior calculus formulation". In: *arXiv preprint arXiv:2303.06082* (2023).

Complex structure of non linear elasticity

$$\begin{tikzcd}
\Omega^0(\mathcal{S};T\mathcal{S}) \arrow[r, "\mathrm{d}^0_{\nabla}"] \arrow[d, "\varphi^*_{\mathrm{f}}"] & \Omega^1(\mathcal{S};T\mathcal{S}) \arrow[r, "\mathrm{d}^1_{\nabla}"] \arrow[d, "\varphi^*_{\mathrm{f}}"] & \Omega^2(\mathcal{S};T\mathcal{S}) \arrow[r, "\mathrm{d}^2_{\nabla}"] \arrow[d, "\varphi^*_{\mathrm{f}}"] & \Omega^3(\mathcal{S};T\mathcal{S}) \arrow[d, "\varphi^*_{\mathrm{f}}"] & \textcolor{red}{\text{(spatial)}} \\
\Omega^0_{\varphi}(\mathcal{B};T\mathcal{S}) \arrow[r, "\tilde{\mathrm{d}}^0_{\tilde{\nabla}}"] \arrow[d, "\varphi^*_{\mathrm{v}}"] & \Omega^1_{\varphi}(\mathcal{B};T\mathcal{S}) \arrow[r, "\tilde{\mathrm{d}}^1_{\tilde{\nabla}}"] \arrow[d, "\varphi^*_{\mathrm{v}}"] & \Omega^2_{\varphi}(\mathcal{B};T\mathcal{S}) \arrow[r, "\tilde{\mathrm{d}}^2_{\tilde{\nabla}}"] \arrow[d, "\varphi^*_{\mathrm{v}}"] & \Omega^3_{\varphi}(\mathcal{B};T\mathcal{S}) \arrow[d, "\varphi^*_{\mathrm{v}}"] & \textcolor{red}{\text{(material)}} \\
\Omega^0(\mathcal{B};T\mathcal{B}) \arrow[r, "\hat{\mathrm{d}}^0_{\hat{\nabla}}"] & \Omega^1(\mathcal{B};T\mathcal{B}) \arrow[r, "\hat{\mathrm{d}}^1_{\hat{\nabla}}"] & \Omega^2(\mathcal{B};T\mathcal{B}) \arrow[r, "\hat{\mathrm{d}}^2_{\hat{\nabla}}"] & \Omega^3(\mathcal{B};T\mathcal{B}) & \textcolor{red}{\text{(convective)}}
\end{tikzcd}$$

Outline

Port-Hamiltonian formulation of linear elastodynamics

Euler Lagrange equations for linear elasticity

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = \mathrm{div} \boldsymbol{\sigma}, \qquad \boldsymbol{\sigma} := \frac{\delta U}{\delta \boldsymbol{\varepsilon}} = \mathcal{K} \boldsymbol{\varepsilon}, \quad \text{Cauchy stress tensor.}$$

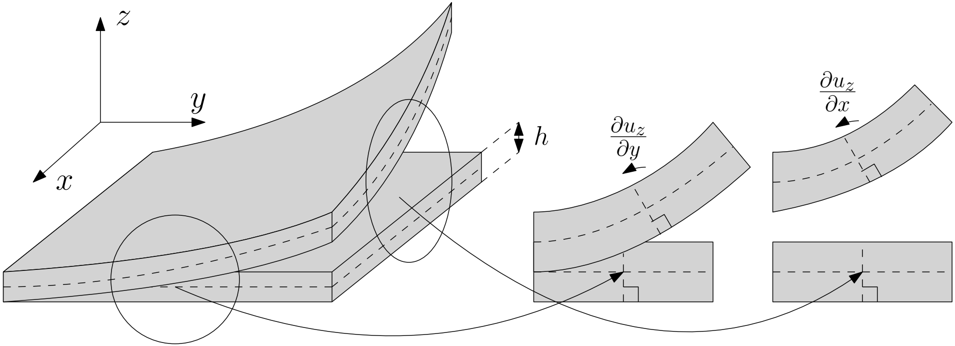
The dynamics of $\boldsymbol{v}$ and $\boldsymbol{\sigma}$ is ruled by the port-Hamiltonian PDE

$$\begin{bmatrix} \rho & 0 \\ 0 & \mathcal{K}^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\sigma} \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{def} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\sigma} \end{pmatrix} \qquad \mathrm{tr}_{\boldsymbol{n}} \boldsymbol{\sigma} = \boldsymbol{u},$$

$$\boldsymbol{y} = \mathrm{tr}\, \boldsymbol{v}.$$

- $\mathrm{tr}\, \boldsymbol{\xi} := \boldsymbol{\xi}|_{\partial\Omega}$ Dirichlet trace;
- $\mathrm{tr}_{\boldsymbol{n}} \boldsymbol{\Xi} := \boldsymbol{\Xi} \cdot \boldsymbol{n}|_{\partial\Omega}$ Normal trace of a tensor.

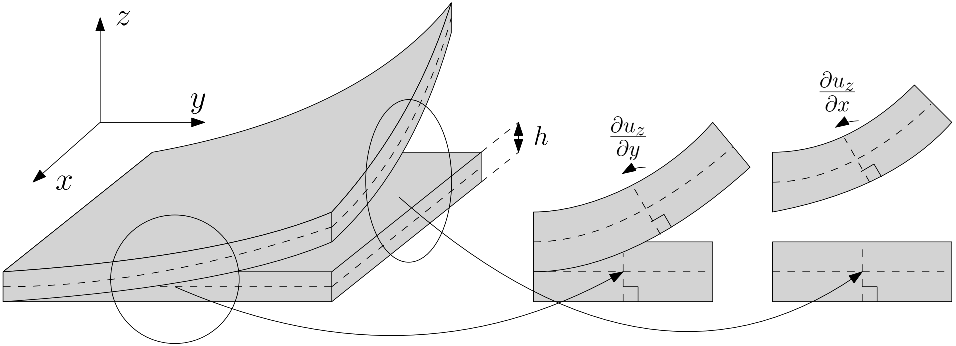
Kirchhoff plate model hypothesis



The fibers (segments perpendicular to the mid-plane before deformation):

- are inextensible.
- remain straight and perpendicular to the middle surface during deformation.

Kirchhoff plate model hypothesis



Displacement field for the bending behavior

$$d_x = -z\partial_x d_z,$$
$$d_y = -z\partial_y d_z,$$
$$d_z = d_z(x, y)$$

Port-Hamiltonian formulation of the Kirchhoff plate

Vertical velocity $v_z(t) = \frac{\partial d_z}{\partial t}$ Bending strain $\boldsymbol{\varepsilon}_b = \text{hess}\, d_z$

Euler Lagrange equations:

$$\rho h \frac{\partial v_z}{\partial t} = -\mathrm{div}\,\mathrm{div} \boldsymbol{\sigma}_b, \qquad \boldsymbol{\sigma}_b := \frac{\delta U}{\delta \boldsymbol{\varepsilon}_b} = \mathcal{K}_b \boldsymbol{\varepsilon}_b,$$

Bending momenta tensor.

The dynamics of v_z and $\boldsymbol{\sigma_b}$ in $\Omega \subset \mathbb{R}^2$

$$\begin{bmatrix} \rho h & 0 \\ 0 & \mathcal{K}_b^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} v_z \\ \boldsymbol{\sigma}_b \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{div}\,\mathrm{div} \\ \text{hess} & 0 \end{bmatrix} \begin{pmatrix} v_z \\ \boldsymbol{\sigma}_b \end{pmatrix}, \qquad \begin{pmatrix} \text{tr}_{1,nn}, \\ \text{tr}_{nn} \end{pmatrix} \boldsymbol{\sigma}_b = \boldsymbol{u},$$

$$\boldsymbol{y} = \begin{pmatrix} \text{tr} \\ \text{tr}_{1,n} \end{pmatrix} v_z$$

- $\text{tr}_{1,n}\, v_z := \partial_{\boldsymbol{n}} v_z|_{\partial\Omega}$ Neumann trace;
- $\text{tr}_{nn}\, \boldsymbol{\sigma}_b := (\boldsymbol{n} \otimes \boldsymbol{n}, \boldsymbol{\sigma}_b)_{\mathrm{F}}|_{\partial\Omega}$ normal to normal trace of a tensor;
- $\text{tr}_{1,nn}\, \boldsymbol{\sigma}_b := -\boldsymbol{n} \cdot \mathrm{div} \boldsymbol{\sigma}_b - \partial_{\boldsymbol{t}}(\boldsymbol{n}^\top \boldsymbol{\sigma}_b \boldsymbol{t})|_{\partial\Omega}$, shear force trace ($\boldsymbol{t}$ tangential direction)

Outline

Dual field formulation and the discrete Hodge star

To construct an **isomorphic Hodge star** dual meshes are needed. Difficulties:

- Constructing **high quality dual meshes is difficult**;
- **Mesh entities** for the dual mesh **lay outside** of the physical domain.

If **one mesh** is used, the weak Hodge is an L^2 **projection between dual FE spaces**[5]

$$(\psi^k, \lambda^k)_{L^2(\Omega)} = \langle \psi^k \,|\mu^{n-k}\rangle_\Omega \qquad \Longrightarrow \qquad \mathbf{M}^k \boldsymbol{\lambda}^k = \mathbf{L}^{n-k} \boldsymbol{\mu}^{n-k}, \qquad \mathbf{L}^{n-k} \text{ rectangular.}$$

C^1 splines allows to construct an isomorphic discrete Hodge star[6].

[5] Hiptmair, "Discrete Hodge operators".

[6] Kapidani and Hernandez, "High Order Geometric Methods With Splines: An Analysis of Discrete Hodge-Star Operators".

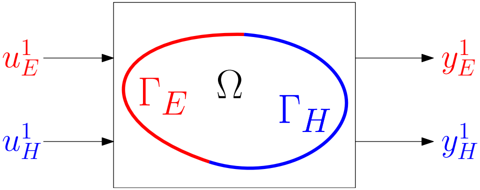
Illustration: 3D Maxwell equation with mixed boundary conditions

$$\begin{pmatrix} \partial_t \boldsymbol{E}^2 \\ \partial_t \boldsymbol{H}^2 \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{curl} \\ -\,\mathrm{curl} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{E}^1 \\ \boldsymbol{H}^1 \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{E}^1 &:= \star \boldsymbol{E}^2 \\ \boldsymbol{H}^1 &:= \star \boldsymbol{H}^2 \end{aligned}$$

Power balance (Dirac structure):

$$\langle \boldsymbol{E}^1 \,|\partial_t \boldsymbol{E}^2\rangle_\Omega + \langle \boldsymbol{H}^1 \,|\partial_t \boldsymbol{H}^2\rangle_\Omega = \langle \mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{E}^1 \,|\, \mathrm{tr}_{\boldsymbol{t}\perp}\, \boldsymbol{H}^1\rangle_{\partial\Omega}.$$

$\mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{\xi} := \boldsymbol{n} \times (\boldsymbol{\xi} \times \boldsymbol{n})|_{\partial\Omega}$ tangential trace;
$\mathrm{tr}_{\boldsymbol{t}\perp}\, \boldsymbol{\xi} := \boldsymbol{n} \times \boldsymbol{\xi}|_{\partial\Omega}$ twisted tangential trace.



Causal (Input-output) behavior:

$$\mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{E}^1|_{\Gamma_E} = u_E^1, \qquad y_E^1 = \mathrm{tr}_{\boldsymbol{t}\perp}\, \boldsymbol{H}^1|_{\Gamma_E},$$
$$\mathrm{tr}_{\boldsymbol{t}\perp}\, \boldsymbol{H}^1|_{\Gamma_H} = u_H^1, \qquad y_H^1 = \mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{E}^1|_{\Gamma_E}.$$

Primal-dual structure of the equations and function spaces

The primal system

$$\begin{pmatrix} \partial_t \boldsymbol{E}^2 \\ \partial_t \boldsymbol{H}^1 \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl}^* & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{E}^2 \\ \boldsymbol{H}^1 \end{pmatrix}$$

The dual system

$$\begin{pmatrix} \partial_t \boldsymbol{E}^1 \\ \partial_t \boldsymbol{H}^2 \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{curl}^* \\ -\mathrm{curl} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{E}^1 \\ \boldsymbol{H}^2 \end{pmatrix},$$

The one forms are conforming $\boldsymbol{E}^1(t),\ \boldsymbol{H}^1(t) \in H^{\mathrm{curl}}(\Omega)$.
The two forms square integrable $\boldsymbol{E}^2(t), \boldsymbol{H}^2(t) \in L^2(\Omega, \mathbb{R}^3)$.

To obtain a discrete Dirac structure a de Rham subcomplex is needed:

- $\boldsymbol{E}^2(t),\ \boldsymbol{H}^2(t) \in H^{\mathrm{div}}(\Omega)$[7];
- Use the broken $H^{\mathrm{div}}(\mathcal{T}_h)$ space[8].

In both cases the Dirac structure hold pointwise $\nabla \times H^{\mathrm{curl}}(\Omega) \subset H^{\mathrm{div}}(\Omega) \hookrightarrow H^{\mathrm{div}}(\mathcal{T}_h)$

[7] Nédélec, "Mixed finite elements in $\mathbb{R}^3$"; Brugnoli, Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using FEEC".
[8] Brugnoli, Rashad, Zhang, et al., "Finite element hybridization of port-Hamiltonian systems".

The discrete formulation

Primal discrete formulation

Find $\boldsymbol{E}_h^1 \in \mathrm{NED}_{\boldsymbol{s}}^1(\Omega)$, $\boldsymbol{H}_h^2 \in \mathrm{RT}_{\boldsymbol{s}}(\mathcal{T}_h)$ such that $\mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{E}_h^1|_{\Gamma_E} = u_{E,h}^1$ and

$$
\begin{aligned}
(\psi_h^1, \partial_t \boldsymbol{E}_h^1)_\Omega &= +(\mathrm{curl}\, \psi_h^1, \boldsymbol{H}_h^2)_\Omega + \langle \mathrm{tr}_{\boldsymbol{t}}\, \psi_h^1 \,|\, u_{H,h}^1 \rangle_{\Gamma_2}, && \forall \psi_h^1 \in \mathrm{NED}_{\boldsymbol{s},\Gamma_E}(\Omega),\\
(\psi_h^2, \partial_t \boldsymbol{H}_h^2)_\Omega &= -(\psi_h^2, \mathrm{curl}\, \boldsymbol{E}_h^1)_\Omega, && \forall \psi_h^2 \in \mathrm{RT}_{\boldsymbol{s}}(\mathcal{T}_h).
\end{aligned}
$$

Algebraic realization

$$
\begin{bmatrix} \mathbf{M}_\Omega^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{T}_h}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\begin{pmatrix} \dot{\mathbf{E}}^1 \\ \dot{\mathbf{H}}^2 \\ \dot{\boldsymbol{\lambda}}_E \end{pmatrix}
=
\begin{bmatrix} \mathbf{0} & (\mathbf{D}^1)^\top & \mathbf{T}_{\Gamma_E}^1 \\ -\mathbf{D}^1 & \mathbf{0} & \mathbf{0} \\ -(\mathbf{T}_{\Gamma_E}^1)^\top & \mathbf{0} & \mathbf{0} \end{bmatrix}
\begin{pmatrix} \mathbf{E}^1 \\ \mathbf{H}^2 \\ \boldsymbol{\lambda}_E \end{pmatrix}
+
\begin{bmatrix} \mathbf{B}_{\Gamma_H}^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}
\begin{pmatrix} \mathbf{u}_H^1 \\ \mathbf{u}_E^1 \end{pmatrix}.
$$

Preservation of the Dirac structure

Discrete power balance

By using both primal and dual system the power balance is retrieved

$$\langle \boldsymbol{E}_h^1 \,|\partial_t \boldsymbol{E}_h^2\rangle_\Omega + \langle \boldsymbol{H}_h^1 \,|\partial_t \boldsymbol{H}_h^2\rangle_\Omega - \langle \mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{E}_h^1 \,|\, \mathrm{tr}_{\boldsymbol{t}\perp}\, \boldsymbol{H}_h^1\rangle_{\partial\Omega} = 0.$$

Proof From the discrete De Rham, it holds

$$\partial_t \boldsymbol{E}_h^2 = \mathrm{curl}\, \boldsymbol{H}_h^1, \qquad \partial_t \boldsymbol{H}_h^2 = -\,\mathrm{curl}\, \boldsymbol{E}_h^1.$$

Take the duality product with $\boldsymbol{E}_h^1$ and $\boldsymbol{H}_h^1$ on each cell K of the mesh, summing up and applying Stokes theorem

$$\sum_{K\in\mathcal{T}_h} \langle \boldsymbol{E}_h^1 \,|\partial_t \boldsymbol{E}_h^2\rangle_K + \langle \boldsymbol{H}_h^1 \,|\partial_t \boldsymbol{H}_h^2\rangle_K = \sum_{K\in\mathcal{T}_h} \langle \mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{E}_h^1 \,|\, \mathrm{tr}_{\boldsymbol{t}\perp}\, \boldsymbol{H}_h^1\rangle_{\partial K}.$$

Since $\boldsymbol{E}_h^1$ and $\boldsymbol{H}_h^1$ are tangential continuous across cells, the inter-cell terms vanish

$$\langle \boldsymbol{E}_h^1 \,|\partial_t \boldsymbol{E}_h^2\rangle_\Omega + \langle \boldsymbol{H}_h^1 \,|\partial_t \boldsymbol{H}_h^2\rangle_\Omega = \langle \mathrm{tr}_{\boldsymbol{t}}\, \boldsymbol{E}_h^1 \,|\, \mathrm{tr}_{\boldsymbol{t}\perp}\, \boldsymbol{H}_h^1\rangle_{\partial\Omega}.$$

Outline

The continuous formulation and the equivalent local problem[9]

Local primal problem

For each cell $K \in \mathcal{T}_h$ find

$$\boldsymbol{E}^1 \in H^{\mathsf{curl}}(K),\ \boldsymbol{H}^2 \in H^{\mathrm{div}}(K) \text{ such that } \tilde{\gamma}_T \boldsymbol{E}^1|_{\partial K} = \boldsymbol{E}^{1,\boldsymbol{t}} \text{ and}$$

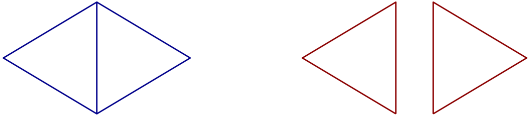
$$\begin{aligned}
(\boldsymbol{\psi}^1, \partial_t \boldsymbol{E}^1)_K &= +(\mathrm{curl}\, \boldsymbol{\psi}^1, \boldsymbol{H}^2)_K, \qquad & \forall \boldsymbol{\psi}^1 \in \tilde{H}^{\mathsf{curl}}_{\partial K}(K), \\
(\boldsymbol{\psi}^2, \partial_t \boldsymbol{H}^2)_K &= -(\boldsymbol{\psi}^2, \mathrm{curl}\, \boldsymbol{E}^1)_K, \qquad & \forall \boldsymbol{\psi}^2 \in H^{\mathrm{div}}(K).
\end{aligned}$$

Main idea: allow $\boldsymbol{E}^{1,\boldsymbol{t}}$ to be an independent variable and set the constraint $\tilde{\gamma}_T \boldsymbol{E}^1|_{\partial K} = \boldsymbol{E}^{1,\boldsymbol{t}}$ via a Lagrange multiplier $\boldsymbol{H}^{1,\boldsymbol{n}}$.

[9] Awanou et al., "Hybridization and postprocessing in finite element exterior calculus"; Brugnoli, Rashad, Zhang, et al., "Finite element hybridization of port-Hamiltonian systems".

Hybrid weak formulation

$$\mathcal{F}_h = \bigcup_{K\in\mathcal{T}_h} \partial K \qquad \mathcal{T}_h := \bigsqcup_{K\in\mathcal{T}_h} \partial K$$



Hybrid primal formulation (Equivalent to the mixed one)

- Local variables $\boldsymbol{E}^1 \in H^{\mathsf{curl}}(\mathcal{T}_h)$, $\boldsymbol{H}^2 \in H^{\mathrm{div}}(\mathcal{T}_h)$, $\boldsymbol{H}^{1,\boldsymbol{n}} \in H^{\mathsf{curl},\boldsymbol{n}}(\partial\mathcal{T}_h)$
- global variable $\boldsymbol{E}^{1,\boldsymbol{t}} \in H^{\mathsf{curl},\boldsymbol{t}}(\mathcal{F}_h)$

such that $\tilde{\gamma}_T \boldsymbol{E}^{1,\boldsymbol{t}}|_{\Gamma_1} = u_1^1$ and

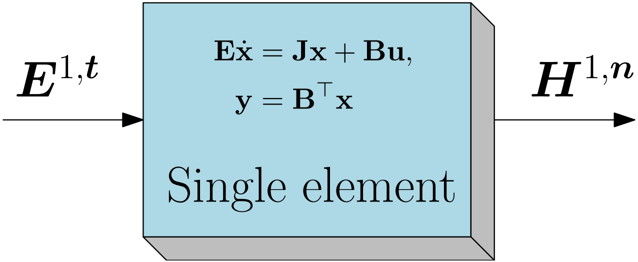
$$
\begin{aligned}
(\boldsymbol{\psi}^1, \partial_t \boldsymbol{E}^1)_{\mathcal{T}_h} &= +(\mathsf{curl}\,\boldsymbol{\psi}^1, \boldsymbol{H}^2)_{\mathcal{T}_h} + \langle \tilde{\gamma}_T \boldsymbol{\psi}^1, \tilde{\gamma}_T \boldsymbol{H}^{1,\boldsymbol{n}} \rangle_{\partial\mathcal{T}_h}, && \forall \boldsymbol{\psi}^1 \in \tilde{H}^{\mathsf{curl}}(\mathcal{T}_h),\\
(\boldsymbol{\psi}^2, \partial_t \boldsymbol{H}^2)_{\mathcal{T}_h} &= -(\boldsymbol{\psi}^2, \mathsf{curl}\,\boldsymbol{E}^1)_{\mathcal{T}_h}, && \forall \boldsymbol{\psi}^2 \in H^{\mathrm{div}}(\mathcal{T}_h),\\
0 &= -\langle \tilde{\gamma}_T \boldsymbol{\psi}^{1,\boldsymbol{n}}, \tilde{\gamma}_T \boldsymbol{E}^1 - \tilde{\gamma}_T \boldsymbol{E}^{1,\boldsymbol{t}} \rangle_{\partial\mathcal{T}_h}, && \boldsymbol{\psi}^{1,\boldsymbol{n}} \in H^{\mathsf{curl},\boldsymbol{n}}(\partial\mathcal{T}_h),\\
0 &= -\langle \tilde{\gamma}_T \boldsymbol{\psi}^{1,\boldsymbol{t}}, \tilde{\gamma}_T \boldsymbol{H}^{1,\boldsymbol{n}} \rangle_{\partial\mathcal{T}_h} + \langle \tilde{\gamma}_T \boldsymbol{\psi}^1 \,|\, u_2^1 \rangle_{\Gamma_2}, && \boldsymbol{\psi}^{1,\boldsymbol{t}} \in H^{\mathsf{curl},\boldsymbol{t}}_{\Gamma_1}(\mathcal{F}_h).
\end{aligned}
$$

Hybridization as transformer interconnection of port-Hamiltonian systems

Start from the local problem where a Lagrange multiplier imposes the essential BC

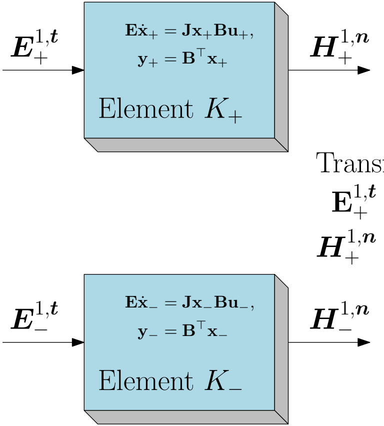
$$\begin{bmatrix} \mathbf{M}_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{x}}_K \\ \dot{\boldsymbol{\lambda}}_{\partial K}^{\boldsymbol{n}} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_K & \mathbf{G}_{\partial K} \\ -\mathbf{G}_{\partial K}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x}_K \\ \boldsymbol{\lambda}_{\partial K}^{\boldsymbol{n}} \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{\partial K} \end{bmatrix} \mathbf{u}_{\partial K}^{\boldsymbol{t}},$$

$$\mathbf{y}_{\partial K}^{\boldsymbol{n}} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_{\partial K}^\top \end{bmatrix} \begin{pmatrix} \mathbf{x}_K \\ \boldsymbol{\lambda}_{\partial K}^{\boldsymbol{n}} \end{pmatrix}.$$



Hybridization as transformer interconnection of port-Hamiltonian systems

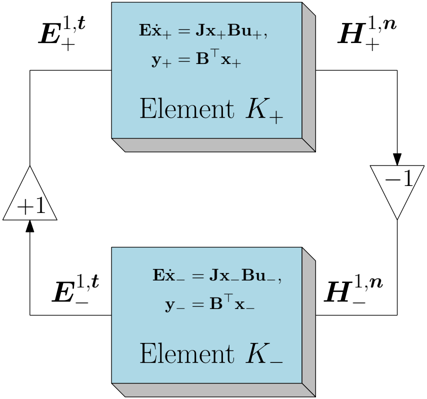
Connect adjacent elements via an energy preserving interconnection.



Transformer

$$\mathbf{E}_+^{1,\boldsymbol{t}} = +\mathbf{E}_-^{1,\boldsymbol{t}}$$

$$\boldsymbol{H}_+^{1,\boldsymbol{n}} = -\boldsymbol{H}_-^{1,\boldsymbol{n}}$$



Two nested pHDAE systems

Given a finite element basis the following system is obtained (both for primal and dual)

$$\begin{bmatrix} \mathbf{E}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{x}}_l \\ \dot{\mathbf{x}}_g \end{pmatrix} = \begin{bmatrix} \mathbf{J}_l & \mathbf{C}_{lg} \\ -\mathbf{C}_{lg}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x}_l \\ \mathbf{x}_g \end{pmatrix} + \begin{bmatrix} \mathbf{B}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_g \end{bmatrix} \begin{pmatrix} \mathbf{u}_l \\ \mathbf{u}_g \end{pmatrix},$$

where $\mathbf{C}_{lg}$ and $\mathbf{B}_l$ are injective.

- the subscript l refers to local quantities;
- the subscript g refers to global quantities.

Time discretization and static condensation

The application of the **implicit midpoint scheme** leads to the saddle point problem

$$\begin{bmatrix} \mathbf{A}_l & -\mathbf{C} \\ \mathbf{C}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x}_l \\ \mathbf{x}_g \end{pmatrix} = \begin{pmatrix} \mathbf{b}_l \\ \mathbf{b}_g \end{pmatrix}, \qquad \mathbf{A} \text{ is block diagonal.}$$

Global problem: coupled trace of the conforming field

Static condensation means construct the Schur complement

$$\mathbf{C}^\top \mathbf{A}_l^{-1} \mathbf{C} \mathbf{x}_g = \mathbf{b}_g - \mathbf{C}^\top \mathbf{A}^{-1} \mathbf{b}_l,$$

$\mathbf{C}^\top \mathbf{A}_l^{-1} \mathbf{C}$ has a positive semidefinite symmetric part.

Parallelizable local problems

The local problem is block diagonal and can be solved in parallel.

$$\mathbf{x}_l = \mathbf{A}^{-1} \mathbf{b}_l + \mathbf{A}^{-1} \mathbf{C} \mathbf{x}_g.$$

Conclusion

Some insights from the port-Hamiltonian perspective:

- System theoretic vision for multi-physical modelling (control/model reduction);
- Emphasis on the boundary conditions and preservation of the power flow;
- Hilbert complex structure $\equiv$ Dirac structure;
- Still much to do for continuum mechanics

Bibliography I

- Awanou, G. et al. “Hybridization and postprocessing in finite element exterior calculus”. In: *Mathematics of Computation* 92 (2023).
- Brugnoli, A., R. Rashad, and S. Stramigioli. “Dual field structure-preserving discretization of port-Hamiltonian systems using FEEC”. In: *Journal of Computational Physics* 471 (2022).
- Brugnoli, A., R. Rashad, Yi. Zhang, et al. “Finite element hybridization of port-Hamiltonian systems”. In: (2023).
- Courant, T. J. “Dirac manifolds”. In: *Transactions of the American Mathematical Society* 319.2 (1990), pp. 631–661.
- Hiptmair, R. “Discrete Hodge operators”. In: *Numerische Mathematik* 90.2 (2001), pp. 265–289.
- Kapidani, Bernard and Rafael Vázquez Hernandez. “High Order Geometric Methods With Splines: An Analysis of Discrete Hodge-Star Operators”. In: *SIAM Journal on Scientific Computing* 44.6 (2022), A3673–A3699.

Bibliography II

- Nédélec, J. C. “Mixed finite elements in $\mathbb{R}^3$”. In: *Numerische Mathematik* 35.3 (1980).
- Rashad, R. et al. “Intrinsic nonlinear elasticity: An exterior calculus formulation”. In: *arXiv preprint arXiv:2303.06082* (2023).
- Schaft, Arjan J. van der and Bernhard Maschke. “Hamiltonian formulation of distributed-parameter systems with boundary energy flow”. In: *Journal of Geometry and Physics* 42 (1-2 2002), pp. 166–194.

Outline

Numerical tests

Propagation of an eigensolution in a cavity

Unit cube domain: $\Omega = [0, 1]^3$.
Boundary sub-partitions:

$$\Gamma_E = \{(x, y, z)|\ x = 0 \cup y = 0 \cup z = 0\}, \qquad \Gamma_H = \{(x, y, z)|\ x = 1 \cup y = 1 \cup z = 1\}.$$

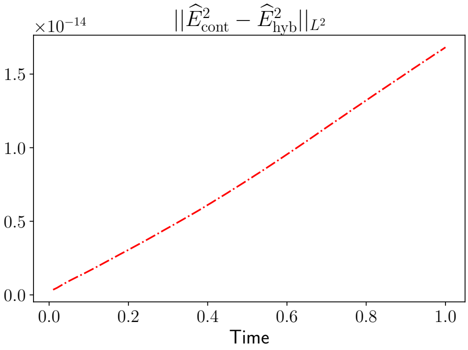
Given the functions

$$\boldsymbol{g}(x, y, z) = \begin{pmatrix} -\cos(x)\sin(y)\sin(z) \\ 0 \\ \sin(x)\sin(y)\cos(z) \end{pmatrix}, \qquad f(t) = \frac{\sin(\sqrt{3}t)}{\sqrt{3}}.$$

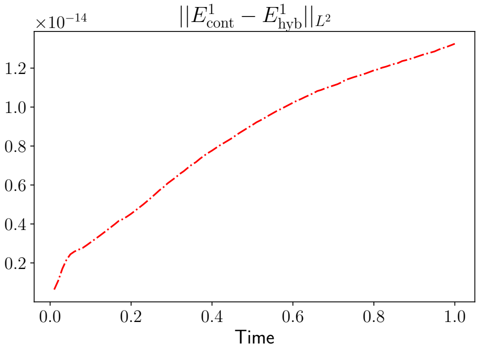
The Maxwell equations are solved by the eigenmode

$$\boldsymbol{E}^2_{\mathrm{ex}} = \star\boldsymbol{g}^\flat \frac{df}{dt}, \qquad \boldsymbol{E}^1_{\mathrm{ex}} = \boldsymbol{g}^\flat \frac{df}{dt},$$
$$\boldsymbol{H}^2_{\mathrm{ex}} = -\,\mathrm{d}\boldsymbol{g}^\flat f, \qquad \boldsymbol{H}^1_{\mathrm{ex}} = -\star\ \mathrm{d}\boldsymbol{g}^\flat f.$$

Results: equivalence continuous and hybrid scheme



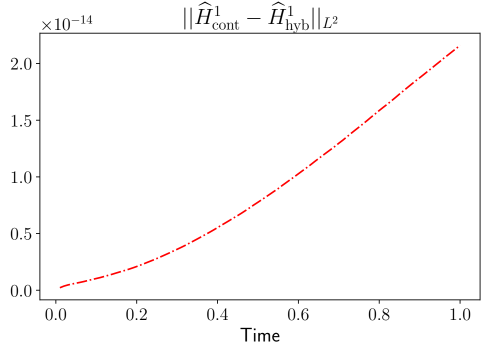
(a) Error $\boldsymbol{E}^2$



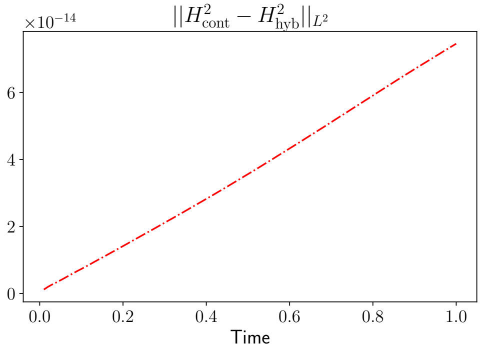
(b) Error $\boldsymbol{E}^1$

L^2 difference continuous and hybrid solution (Electric field).

Results: equivalence continuous and hybrid scheme



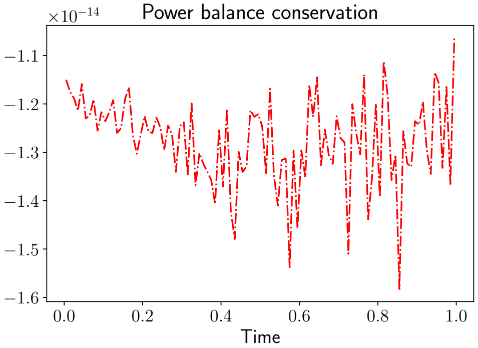
(a) Error $\boldsymbol{H}^1$



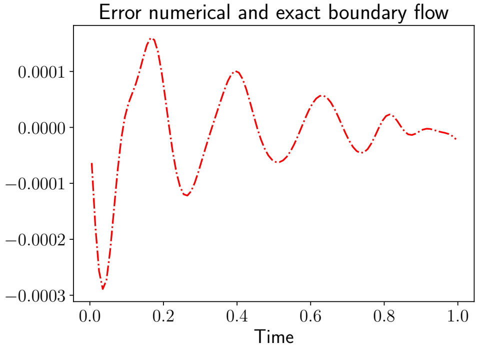
(b) Error $\boldsymbol{H}^2$

L^2 difference continuous Galerkin and hybrid solution (Magnetic field).

Results: conservation properties



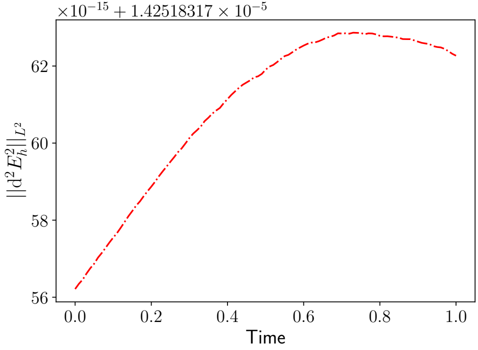
(a) $\dot{H}_h = < \widehat{y}^1_{E,h} | u^1_{E,h} >_{\partial M} + < \widehat{u}^1_{H,h} | y^1_{H,h} >_{\partial M}$



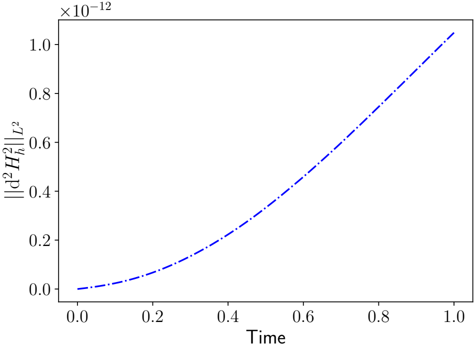
(b) Error exact and interpolated boundary flow

Power balance (left) and error on the power flow (right).

Results: conservation properties



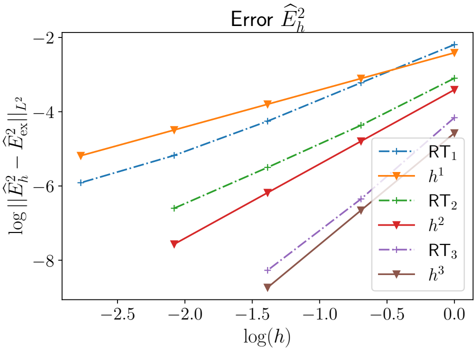
(a) L^2 norm of $\mathrm{d}\boldsymbol{E}_h^2$



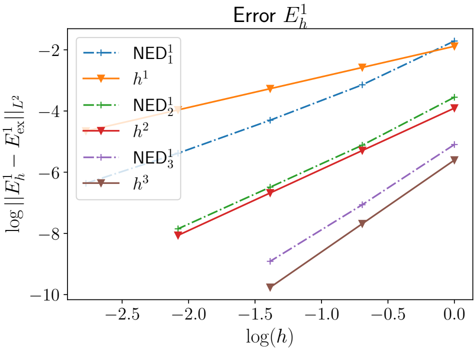
(b) L^2 norm of $\mathrm{d}\boldsymbol{H}_h^2$

L^2 norm divergence of the two forms $\boldsymbol{E}_h^2, \boldsymbol{H}_h^2$.

Convergence rate



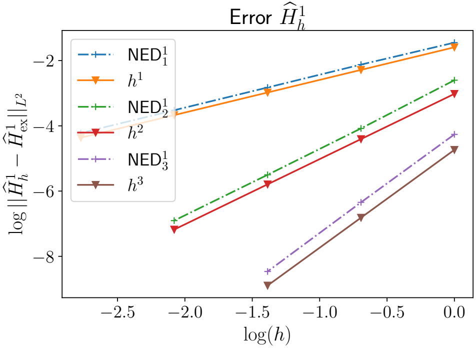
(a) L^2 error for $\boldsymbol{E}_h^2$



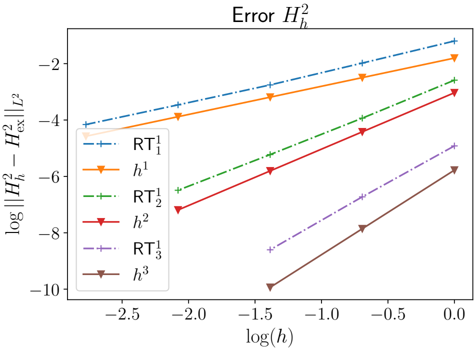
(b) L^2 error for $\boldsymbol{E}_h^1$

Convergence rate Electric field computed via the L^2 norm.

Convergence rate



(a) L^2 error for $\boldsymbol{H}_h^1$



(b) L^2 error for $\boldsymbol{H}_h^2$

Convergence rate Electric field computed via the L^2 norm.

Size reduction achieved by the hybridization

Pol. Degree s	N_{elem}	N° dofs. continuous	N° dofs. hybrid	$N_{\mathrm{hyb}}/N_{\mathrm{cont}}$
1	1	43	19	44 %
	2	290	98	38%
	4	2140	604	28%
	8	16472	4184	25%
	16	129328	31024	24%
2	1	164	74	45%
	2	1156	436	37%
	4	8696	2936	33%
	8	67504	21424	32%
3	1	399	165	41%
	2	2886	1014	35%
	4	21972	6996	32%