Linearly-implicit energy-momentum preserving scheme for geometrically nonlinear mechanics

A mixed finite element approach

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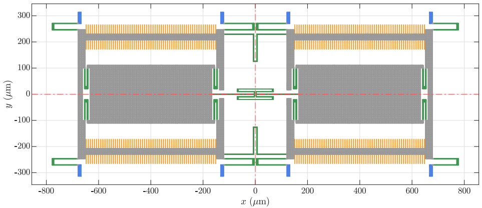
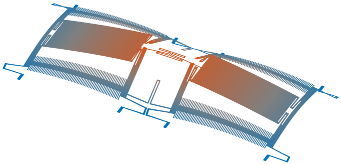
Summary

Numerical modelling of geometrical nonlinearities

Geometrical nonlinearities: arise from large displacements; occur in aeronautics, wind energy, musical acoustics, MEMS.
Simulation is crucial: animation, sound synthesis, control and optimization.



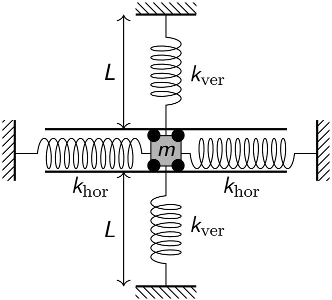
(a)



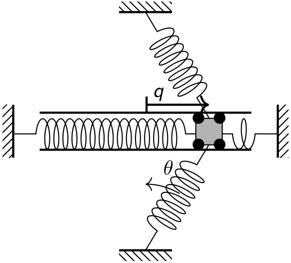
MEMS device (left) and First mode (right) Schiwietz et al., “Shape optimization of geometrically nonlinear modal coupling coefficients: an application to MEMS gyroscopes”

Summary

Duffing oscillator



Undeformed configuration



Deformed configuration

The material behavior of the springs is linear elastic (only geometrical nonlinearities)

$$m\ddot{q} = -2k_{\mathrm{hor}}q - 2k_{\mathrm{ver}}\delta \sin(\theta),$$

- $\delta = \sqrt{L^2 + q^2} - L$ is the elongation of the vertical springs;
- $\sin\theta = q/\sqrt{L^2 + q^2}$

Cubic approximation and canonical Hamiltonian form

Consider a Taylor expansion

$$(L^2 + q^2)^{-1/2} = \frac{1}{L} - \frac{q^2}{2L^3} + \mathcal{O}(q^4).$$

Then the Duffing oscillator is obtained

$$\ddot{q} = -\alpha q - \beta q^3, \qquad \alpha := 2k_{\mathrm{hor}}/m, \quad \beta := k_{\mathrm{ver}}/(mL^3).$$

Canonical Hamiltonian equations ($p := \dot{q}$):

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix}.$$

The Hamiltonian is given by

$$H(q, \dot{q}) = \frac{1}{2}\dot{q} + \frac{1}{2}\alpha q^2 + \frac{1}{4}\beta q^4.$$

Poisson formulation of a Hamiltonian system

A general Hamiltonian system can be written as

$$\dot{x} = J(x)\nabla H(z), \qquad x \in \mathbb{R}^n, \qquad J(x) = -J(x)^\top.$$

Define the **Poisson bracket** for smooth functions $F, G : \mathbb{R}^n \to \mathbb{R}$ as

$$\{F, G\} := (\nabla F)^\top J(x) \nabla G.$$

Then the system can equivalently be written in Poisson form:

$$\dot{x}_i = \{x_i, H\}, \quad i = 1, \dots, n, \qquad \frac{dF}{dt} = \{F, H\}.$$

Properties:

- Bilinearity: $\{aF + bG, H\} = a\{F, H\} + b\{G, H\}$,
- Skew-symmetry: $\{F, G\} = -\{G, F\}$,
- Jacobi identity: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$,
- Energy conservation: $\frac{dH}{dt} = \{H, H\} = 0$.

Energy preservation via Discrete Gradient method

Energy preservation can be achieved using a discrete gradient.
A discrete gradient $\overline{\nabla} H(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping that satisfies:

1. **Discrete chain rule:** $H(y) - H(x) = \overline{\nabla} H(x, y)^\top (y - x)$,
2. **Consistency:** $\overline{\nabla} H(x, x) = \nabla H(x)$.

Examples:

- Mean Value Discrete Gradient

$$\overline{\nabla}_{\mathrm{MV}} H(x, y) = \int_0^1 \nabla H((1 - \theta)x + \theta y) \, d\theta.$$

- Gonzales Midpoint Discrete Gradient

$$\overline{\nabla}_{\mathrm{G}} H(x, y) = \nabla H\Big(\frac{x + y}{2}\Big) + \frac{H(y) - H(x) - \nabla H(\frac{x+y}{2})^\top (y - x)}{\|y - x\|^2} (y - x), \quad y \neq x.$$

Time stepping

Given a discrete gradient $\overline{\nabla} H$, the corresponding time-stepping method is

$$\frac{x^{n+1} - x^n}{\Delta t} = J\left(\tfrac{x^n + x^{n+1}}{2}\right) \overline{\nabla} H(x^n, x^{n+1}).$$

This one-step method preserves H exactly.

Disadvantages

- **Computational cost:** Implicit methods, Newton solver required. Mean value gradients require numerical quadrature; midpoint gradients require multiple function evaluations.
- **Order of accuracy:** Discrete gradient methods are second-order accurate. Constructing higher-order energy-preserving schemes is challenging[1].

[1] Eidnes, "Order theory for discrete gradient methods"

Towards a Poisson formulation: energy quadratisation

The Hamiltonian

$$H = \frac{1}{2}v^2 + \frac{1}{2}\alpha q^2 + \frac{1}{4}\beta q^4, \qquad v := \dot{q}$$

can be rewritten using the stresses. The strains are given by

$$\varepsilon_1 = q, \quad \varepsilon_2 = q^2$$

The stresses are given by (Legendre transform conjugate variables)

$$\sigma_1 = \alpha\varepsilon_1, \qquad \sigma_2 = \frac{\beta}{2}\varepsilon_2$$

The energy is now rewritten as a quadratic form

$$H = \frac{1}{2}\begin{pmatrix} v \\ \sigma_1 \\ \sigma_2 \end{pmatrix}^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & 2/\beta \end{bmatrix} \begin{pmatrix} v \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \frac{1}{2}\mathbf{x}^\top \mathbf{H}\mathbf{x}.$$

Poisson structure

Idea: introduce the dynamics of the stress variables. The augmented system reads

$$\dot{q} = \mathbf{e}_1^\top \mathbf{x},$$
$$\mathbf{H}\dot{\mathbf{x}} = \mathbf{J}(q)\mathbf{x},$$

where $\mathbf{e}_1 = [1 \quad 0 \quad 0]^\top$ is the first element of the Euclidean canonical basis, and

$$\mathbf{H} = \mathrm{Diag}\begin{bmatrix} 1 \\ 1/\alpha \\ 2/\beta \end{bmatrix}, \quad \mathbf{J}(q) = \begin{bmatrix} 0 & -1 & -2q \\ 1 & 0 & 0 \\ 2q & 0 & 0 \end{bmatrix}.$$

Summary

A general abstract framework

Geometrically nonlinear mechanical models admit an analogous non-canonical Hamiltonian form:

$$\partial_t \boldsymbol{q} = \boldsymbol{v},$$
$$\mathcal{H}\partial_t \boldsymbol{x} = \mathcal{J}(\boldsymbol{q})\boldsymbol{x},$$

with state $\boldsymbol{x} = (\boldsymbol{v}, \boldsymbol{S})$, $\quad \mathcal{H} = \mathrm{Diag}[\rho,\ \mathbb{C}]$

- $\boldsymbol{q}$: displacement, $\quad \boldsymbol{v}$: velocity, $\quad \boldsymbol{S}$: stress-like variable,
- ρ: density, $\quad \mathbb{C}$: compliance tensor.

Poisson operator:

$$\mathcal{J}(\boldsymbol{q}) = \begin{bmatrix} 0 & -\mathcal{L}^*(\mathcal{D}\boldsymbol{q}) \\ \mathcal{L}(\mathcal{D}\boldsymbol{q}) & 0 \end{bmatrix}.$$

Here $\mathcal{D}\boldsymbol{q}$ is a parameter and $\mathcal{L}$ designates the formal adjoint of $\mathcal{L}$

Energy and Variational Basis

Skew-adjointness of $\mathcal{J}(\boldsymbol{q})$ ensures conservation of

$$H(\boldsymbol{v}, \boldsymbol{S}) = \frac{1}{2}\int_\Omega \rho||\boldsymbol{v}||^2 + \boldsymbol{S}\cdot\mathbb{C}\boldsymbol{S}\,\mathrm{d}\Omega, \qquad \dot{H} = 0.$$

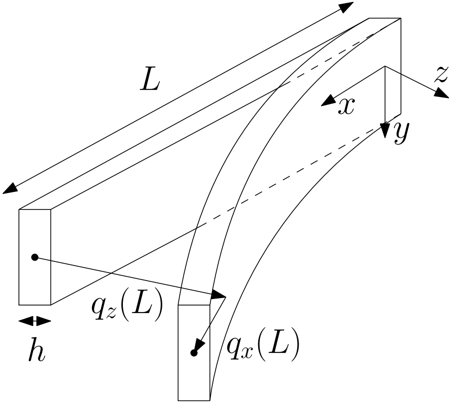
This system is derived from the least action principle $\delta S = 0$ using the Reissner expression of the deformation energy V_R:

$$\begin{aligned} S(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{S}) &:= \int_0^{t_f}\int_\Omega \dot{\boldsymbol{q}}\cdot\rho\boldsymbol{v}\mathrm{d}\Omega\mathrm{d}t - \int_0^{t_f} T(\boldsymbol{v}) + V_R(\boldsymbol{q}, \boldsymbol{S})\mathrm{d}t, \\ T(\boldsymbol{v}) &:= \frac{1}{2}\int_\Omega \rho||\boldsymbol{v}||^2\mathrm{d}\Omega, \\ V_R(\boldsymbol{q}, \boldsymbol{S}) &:= \int_\Omega \boldsymbol{S}\cdot\boldsymbol{E}(\boldsymbol{q}) - \frac{1}{2}\boldsymbol{S}\cdot\mathbb{C}\boldsymbol{S}\mathrm{d}\Omega, \end{aligned}$$

where $\boldsymbol{S}\cdot\boldsymbol{E}$ denotes the tensor contraction and $\boldsymbol{E}$ is a geometrically non linear strain. The formulation is local and requires boundary conditions.

Summary

Von Kármán beams



Assumptions:

- Out-of-plane deflection comparable to thickness
- Axial stretching terms negligible vs. rotation squares

$\Rightarrow$ Quadratic bending–membrane coupling term retained

Strain and energies

The axial strain and the (linearized) curvature are given by

$$\varepsilon_a := \frac{\partial q_x}{\partial x} + \frac{1}{2}\left(\frac{\partial q_z}{\partial x}\right)^2, \qquad \kappa := \frac{\partial^2 q_z}{\partial x^2}.$$

Total strain $\varepsilon = \varepsilon_a + \kappa$. Consider the kinetic and potential energy

$$T = \frac{1}{2}\int_0^L \rho A (v_x^2 + v_z^2)\mathrm{d}x,$$

$$V_R = \int_0^L \left\{ N\varepsilon_a(q_x, q_z) + M\kappa(q_z) - \frac{1}{2EA}N^2 - \frac{1}{2EI}M^2 \right\} \mathrm{d}x.$$

Euler Lagrange equations

Kinematics	**Dynamics**	**Constitutive laws**
$\partial_t q_x = v_x,$	$\rho A\, \partial_t v_x = \partial_x N,$	$(EA)^{-1} N = (\partial_x q_x + 1/2(\partial_x q_z)^2),$
$\partial_t q_z = v_z,$	$\rho A\, \partial_t v_z = -\partial_{xx}^2 M + \partial_x (N\, \partial_x q_z),$	$(EI)^{-1} M = \partial_{xx} q_z.$

Poisson system

The time derivative of the stress variables gives

$$
\begin{aligned}
C_a\,\partial_t N &= \partial_x v_x + (\partial_x q_z)\,\partial_x v_z, && C_a := (EA)^{-1},\\
C_b\,\partial_t M &= \partial_{xx} v_z, && C_b := (EI)^{-1}.
\end{aligned}
$$

The Poisson system reads

$$
\begin{aligned}
\partial_t q_x &= v_x,\\
\partial_t q_z &= v_z,
\end{aligned}
$$

$$
\mathrm{Diag}\begin{bmatrix}\rho A\\ \rho A\\ C_a\\ C_b\end{bmatrix}\frac{\partial}{\partial t}\begin{pmatrix}v_x\\ v_z\\ N\\ M\end{pmatrix} = \begin{bmatrix}0 & 0 & \partial_x & 0\\ 0 & 0 & \partial_x(\circ\,\partial_x q_z) & -\partial_{xx}^2\\ \partial_x & (\partial_x q_z)\partial_x\circ & 0 & 0\\ 0 & \partial_{xx}^2 & 0 & 0\end{bmatrix}\begin{pmatrix}v_x\\ v_z\\ N\\ M\end{pmatrix}.
$$

For this example, the operator $\mathcal{L}(\partial_x q_z)$ and its adjoint are given by

$$
\mathcal{L}(\partial_x q_z) = \begin{bmatrix}\partial_x & (\partial_x q_z)\partial_x\circ\\ 0 & \partial_{xx}^2\end{bmatrix}, \qquad \mathcal{L}^*(\partial_x q_z) = -\begin{bmatrix}\partial_x & 0\\ \partial_x(\circ\partial_x q_z) & -\partial_{xx}^2\end{bmatrix}.
$$

Von Kármán plate model: Strain and energies

The axial strain and curvature generalize the previous model

$$\boldsymbol{\varepsilon}_m = \mathrm{def}(\boldsymbol{q}_m) + \frac{1}{2}\nabla q_z \otimes \nabla q_z, \qquad \boldsymbol{\kappa} = \mathrm{Hess}\, q_z.$$

- The displacement vector $\boldsymbol{q}$ has been split into the membrane displacement $\boldsymbol{q}_m = (q_x \quad q_y)^\top$ and the out-of-plane component q_z;
- $\mathrm{def} := \frac{1}{2}(\nabla + \nabla^\top)$ is the symmetric gradient;
- $\otimes$ is the dyadic product of the two vectors, i.e. $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^\top$;

Consider the kinetic and Reissner deformation energy

$$T = \frac{1}{2}\int_0^L \rho h(||\boldsymbol{v}_m||^2 + ||v_z||^2)\mathrm{d}x,$$
$$V_R = \int_0^L \left\{\boldsymbol{N} \cdot \boldsymbol{\varepsilon}_m(\boldsymbol{q}_m, q_z) + \boldsymbol{M} \cdot \boldsymbol{\kappa}(q_z) - \frac{1}{2}\boldsymbol{N} \cdot \mathbb{C}_m \boldsymbol{N} - \frac{1}{2}\boldsymbol{M} \cdot \mathbb{C}_m \boldsymbol{M}\right\} \mathrm{d}x.$$

Poisson Structure

Applying the generalized Hamilton principle and taking the derivative of the membrane and bending stress tensors $\boldsymbol{N}$, $\boldsymbol{M}$

$$
\begin{aligned}
\partial_t \boldsymbol{q}_m &= \boldsymbol{v}_m, \\
\partial_t q_z &= v_z, \\
\mathrm{Diag}\begin{bmatrix} \rho h \\ \rho h \\ \boldsymbol{C}_m \\ \boldsymbol{C}_b \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{v}_m \\ v_z \\ \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix} &= \begin{bmatrix} 0 & 0 & \mathrm{div} & 0 \\ 0 & 0 & \mathcal{C}(q_z) & -\mathrm{div}\,\mathrm{div} \\ \mathrm{def} & -\mathcal{C}^*(q_z) & 0 & 0 \\ 0 & \mathrm{Hess} & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{v}_m \\ v_z \\ \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}.
\end{aligned}
$$

The operator $\mathcal{C}(q_z)(\cdot)$ acting on symmetric tensors is defined by

$$\mathcal{C}(q_z)(\boldsymbol{N}) := \mathrm{div}(\boldsymbol{N}\,\nabla q_z).$$

Its adjoint is given by

$$\mathcal{C}(q_z)^*(\cdot) = -\mathrm{sym}\left[\nabla(\cdot) \otimes \nabla(q_z)\right].$$

Von Kármán in Airy form: the 2D elasticity complex

The model can be simplified by neglecting the membrane inertia $\rho h \partial_t \boldsymbol{v}_m = 0$, leading to

$$\mathrm{div}\, \boldsymbol{N} = 0.$$

For a simply connected domain, the 2D elasticity complex is exact

$$\mathbb{P}_1 \xrightarrow{\subset} C^\infty \xrightarrow{\mathrm{Air}} C^\infty \otimes \mathbb{S} \xrightarrow{\mathrm{div}} C^\infty \otimes \mathbb{V}$$

where $\mathbb{P}_1 := \mathbb{R} + \boldsymbol{x}^\perp \cdot \mathbb{R}^2$ is the space of first order polynomial. So the membrane stress field can be represented by an Airy potential

$$\boldsymbol{N} = \mathrm{Air}\, \varphi.$$

Plugging this expression in the dynamics

$$\mathrm{div}(\boldsymbol{N}\, \nabla q_z) = \mathrm{Air}\, \varphi \cdot \mathrm{Hess}\, q_z, \qquad \text{since } \mathrm{div}\, \mathrm{Air} \equiv 0.$$

The adjoint complex

$$\mathcal{B}(f, g) := \operatorname{Air} f \cdot \operatorname{Hess} g.$$

The following properties of the bilinear form have been proven[2]

- Symmetry: $\mathcal{B}(f, g) = \mathcal{B}(g, f)$;
- Self-adjointness: $(\mathcal{B}(f, g), h)_\Omega = (g, \mathcal{B}(f, h))_\Omega$ (function f is a parameter).

The expression of the membrane stress still contains the in-plane displacement

$$\boldsymbol{C}_m \operatorname{Air} \varphi = \operatorname{def} \boldsymbol{q}_m + \frac{1}{2} \nabla q_z \otimes \nabla q_z.$$

The idea is to exploit the relation

$$\operatorname{rot} \operatorname{rot} \operatorname{def} = 0$$

from the adjoint complex

$$\mathbf{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{V} \xrightarrow{\operatorname{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\operatorname{rot} \operatorname{rot}} C^\infty$$

[2] Bilbao, "A family of conservative finite difference schemes for the dynamical von Kármán plate equations"

Elimination of $\boldsymbol{q}_m$, $\boldsymbol{v}_m$

Applying rot rot to the inplane

$$\operatorname{rot}\operatorname{rot} \boldsymbol{C}_m \operatorname{Air} \varphi = \frac{1}{2} \operatorname{rot}\operatorname{rot}(\nabla q_z \otimes \nabla q_z).$$

The operator $\operatorname{rot}\operatorname{rot} \boldsymbol{C}_m \operatorname{Air} = \mathrm{Air}^* \boldsymbol{C}_m \operatorname{Air}$ is self-adjoint. A little algebra provides

$$(\mathrm{Air}^* \boldsymbol{C}_m \operatorname{Air}) \varphi = \frac{1}{2} \operatorname{rot}\operatorname{rot}(\nabla q_z \otimes \nabla q_z) = -\frac{1}{2} \mathcal{B}(q_z, q_z).$$

Taking the time derivative provides the dynamics of the Airy potential φ

$$(\mathrm{Air}^* \boldsymbol{C}_m \operatorname{Air}) \frac{\partial \varphi}{\partial t} = -\mathcal{B}(q_z, v_z).$$

Poisson formulation of the Airy form

$$
\begin{aligned}
\partial_t q_z &= v_z, \\
\mathrm{Diag}\begin{bmatrix} \rho h \\ \mathrm{Air}^* \boldsymbol{C}_m \,\mathrm{Air} \\ \boldsymbol{C}_b \end{bmatrix} \frac{\partial}{\partial t}\begin{pmatrix} v_z \\ \varphi \\ \boldsymbol{M} \end{pmatrix} &= \begin{bmatrix} 0 & \mathcal{B}(q_z, \circ) & -\operatorname{div}\operatorname{div} \\ -\mathcal{B}(q_z, \circ) & 0 & 0 \\ \mathrm{Hess} & 0 & 0 \end{bmatrix}\begin{pmatrix} v_z \\ \varphi \\ \boldsymbol{M} \end{pmatrix}.
\end{aligned}
$$

A self-adjoint differential operator $\mathcal{H}$ takes the place of an algebraic energy matrix (implicit Hamiltonian system[3]).

The differential operator and its adjoint read

$$
\mathcal{L}(\mathrm{Air}\, q_z) = \begin{bmatrix} -\mathcal{B}(q_z, \circ) \\ \mathrm{Hess} \end{bmatrix}, \qquad \mathcal{L}^*(\mathrm{Air}\, q_z) = -\begin{bmatrix} \mathcal{B}(q_z, \circ) & -\operatorname{div}\operatorname{div} \end{bmatrix}.
$$

[3]Bendimerad-Hohl et al., "Stokes-Lagrange and Stokes-Dirac representations of N-dimensional port-Hamiltonian systems for modeling and control"

Summary

Canonical Hamiltonian structure for continuum mechanics

Given a solid body $\mathcal{B} \subset \mathbb{R}^3$ and an embedding

$$\boldsymbol{x} = \varphi(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{q}(\boldsymbol{X}, t) \in \mathcal{S} \subset \mathscr{A} = \mathbb{R}^3,$$

the dynamics of a deformable body can be written in canonical Hamiltonian form as

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\widetilde{\boldsymbol{\pi}}} \end{pmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{pmatrix} \delta_{\varphi} H \\ \delta_{\widetilde{\boldsymbol{\pi}}} H \end{pmatrix},$$

where the Hamiltonian is given by

$$H(\widetilde{\boldsymbol{\pi}}, \boldsymbol{F}) = \int_{\mathcal{B}} \frac{||\widetilde{\boldsymbol{\pi}}||^2}{2\rho} + \widetilde{\rho} W(\boldsymbol{F}) \mathrm{d}\boldsymbol{X}, \qquad W \text{ hyperelastic potential}$$

The quantity $\widetilde{\boldsymbol{\pi}} := \widetilde{\rho}\widetilde{\boldsymbol{v}}$ is the material representation linear momentum (and velocity)

$$\widetilde{\boldsymbol{v}}(\boldsymbol{X}, t) : \boldsymbol{X} \in \mathcal{B} \rightarrow T_{\varphi(\boldsymbol{X}, t)} \mathcal{S}$$

Rate of change strain tensor (cartesian coordinates)

To ensure frame indifference the deformation energy must depend on the right Cauchy strain tensor or equivalently the Green Lagrange strain

$$\boldsymbol{E}(\boldsymbol{X}) : T_{\boldsymbol{X}}\mathcal{B} \to T_{\boldsymbol{X}}\mathcal{B}, \qquad \boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I}), \qquad \boldsymbol{F} = \boldsymbol{I} + \nabla \boldsymbol{q}.$$

The kinetic and potential energies are given by

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{B}} \rho ||\widetilde{\boldsymbol{v}}||^2 \mathrm{d}\boldsymbol{X}, \\ V_R &= \int_{\Omega} \boldsymbol{S} \cdot \boldsymbol{E}(\boldsymbol{q}) - \frac{1}{2}\boldsymbol{S} \cdot \mathbb{C}\boldsymbol{S} \, \mathrm{d}\Omega. \end{aligned}$$

Euler Lagrange equations

$$\partial_t \boldsymbol{q} = \widetilde{\boldsymbol{v}}, \qquad \rho\, \partial_t \widetilde{\boldsymbol{v}} = \mathrm{div}(\boldsymbol{F}\boldsymbol{S}), \qquad \mathbb{C}\boldsymbol{S} = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I}).$$

Poisson structure

The rate of change of the Green Lagrange strain is given by[4]

$$\mathbb{C}\partial_t \boldsymbol{S} = \mathrm{sym}(\boldsymbol{F}^\top \nabla \widetilde{\boldsymbol{v}}).$$

Dynamical system[5]

$$\begin{aligned}
\partial_t \boldsymbol{q} &= \widetilde{\boldsymbol{v}}, \\
\begin{bmatrix} \rho & 0 \\ 0 & \boldsymbol{C} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \widetilde{\boldsymbol{v}} \\ \boldsymbol{S} \end{pmatrix} &= \begin{bmatrix} 0 & \mathrm{div}(\boldsymbol{F}\,\circ) \\ \mathrm{sym}(\boldsymbol{F}^\top \nabla\,\circ) & 0 \end{bmatrix} \begin{pmatrix} \widetilde{\boldsymbol{v}} \\ \boldsymbol{S} \end{pmatrix},
\end{aligned}$$

The differential operator and its adjoint read

$$\mathcal{L}(\nabla \boldsymbol{q}) = \mathrm{sym}(\boldsymbol{F}^\top \nabla\,\circ), \qquad \mathcal{L}^*(\nabla \boldsymbol{q}) = -\,\mathrm{div}(\boldsymbol{F}\,\circ).$$

[4] Marsden and Hughes, Mathematical foundations of elasticity

[5] Thoma, Kotyczka, and Egger, "On the velocity-stress formulation for geometrically nonlinear elastodynamics and its structure-preserving discretization"

Summary

Von Kármán beams: Finite element spaces

Denote by E a generic element in an interval mesh $\mathcal{I}_h$.
Velocity (and displacement) space $V_h = \mathrm{CG}_1 \times \mathrm{Her}$:

$$\mathrm{CG}_1 = \{v_h \in C^0([0, L]),\ v_h|_E \in \mathbb{P}_1\},$$
$$\mathrm{Her} = \{v_h \in C^1([0, L]),\ v_h|_E \in \mathbb{P}_3\}.$$

Stress space $\Sigma_h = \mathrm{DG}_4 \times \mathrm{DG}_1$

$$\mathrm{DG}_k = \{v_h|_E \in \mathbb{P}_k,\ \forall E \in \mathcal{I}_h\}$$

The reason for DG_4 comes from the axial resultant dynamics

$$C_a \partial_t N = \partial_x v_x + \partial_x q_z \partial_x v_z$$

$$q_z, v_z \in \mathrm{Her} \ \Rightarrow\ \partial_x q_z \partial_x v_z \in \mathrm{CG}_4,$$
$$v_x \in \mathrm{CG}_1 \ \Rightarrow\ \partial_x v_x \in \mathrm{DG}_0$$

Since $\mathrm{CG}_4 + \mathrm{DG}_0 \subset \mathrm{DG}_4$, choosing a quartic DG space avoids locking.

Weak formulation

Integration by parts of the lines of the linear momentum balance.

$$
\begin{aligned}
\partial_t q_{z,h} &= v_{z,h}, \\
(\psi_x,\, \rho A\, \partial_t v_{x,h}) &= -(\partial_x \psi_x,\, N_h), \\
(\psi_z,\, \rho A\, \partial_t v_{z,h}) &= -(\partial_x \psi_z\, \partial_x q_{z,h},\, N_h) - (\partial_{xx} \psi_z,\, M_h), \\
(\psi_N,\, C_a \partial_t N_h) &= (\psi_N,\, \partial_x q_z^h\, \partial_x v_z^h + \partial_x v_{x,h}), \\
(\psi_M,\, C_b \partial_t M_h) &= (\psi_M,\, \partial_{xx} v_z^h).
\end{aligned}
$$

Integration by parts of the lines of the linear momentum balance.

$$
\begin{gathered}
\dot{\mathbf{q}}_z = \mathbf{v}_z, \\
\mathrm{Diag}\begin{bmatrix} \mathbf{M}_{\rho A} \\ \mathbf{M}_{\rho A} \\ \mathbf{M}_{C_a} \\ \mathbf{M}_{C_b} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{v}_x \\ \mathbf{v}_z \\ \mathbf{n} \\ \mathbf{m} \end{pmatrix} = \begin{bmatrix} 0 & 0 & -\mathbf{D}_{\partial_x}^\top & 0 \\ 0 & 0 & -\mathbf{L}^\top(\mathbf{q}_z) & -\mathbf{D}_{\partial_{xx}}^\top \\ \mathbf{D}_{\partial_x} & \mathbf{L}(\mathbf{q}_z) & 0 & 0 \\ 0 & \mathbf{D}_{\partial_{xx}} & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{v}_x \\ \mathbf{v}_z \\ \mathbf{n} \\ \mathbf{m} \end{pmatrix}
\end{gathered}
$$

Nonlinear elasticity: Finite element spaces

Denote with T a generic cell of the computational mesh $\mathcal{T}_h$.
Velocity (and displacement) space $V_h = \mathrm{CG}_1 \otimes \mathbb{R}^d$:

$$\mathrm{CG}_1 \otimes \mathbb{R}^d = \{v_h \in C^0(\Omega),\ v_h|_T \in \mathbb{P}_1 \otimes \mathbb{R}^d\},$$

Stress space $\Sigma_h = \mathrm{DG}_0 \otimes \mathbb{R}^{d\times d}_{\mathrm{sym}}$. The second Piola stress dynamics is

$$\mathbb{C}\partial_t \boldsymbol{S} = \mathrm{sym}(\boldsymbol{F}^\top \nabla \widetilde{\boldsymbol{v}}).$$

Hence $\boldsymbol{S} \propto \nabla \widetilde{\boldsymbol{v}} \nabla \boldsymbol{q}$.
Since $\nabla \widetilde{\boldsymbol{v}} \in \mathrm{DG}_0,\ \nabla \boldsymbol{q} \in \mathrm{DG}_0$, it holds $\boldsymbol{S} \in \mathrm{DG}_0$.

Weak formulation

$$
\begin{aligned}
\partial_t \boldsymbol{q}_h &= \widetilde{\boldsymbol{v}}_h, \\
(\psi,\, \rho\, \partial_t \widetilde{\boldsymbol{v}}_h)_\Omega &= -(\boldsymbol{F}_h^\top \nabla \psi,\, \boldsymbol{S}_h)_\Omega, \\
(\boldsymbol{\Psi},\, \boldsymbol{C}\, \partial_t \boldsymbol{S}_h)_\Omega &= +(\boldsymbol{\Psi},\, \boldsymbol{F}_h^\top \nabla \widetilde{\boldsymbol{v}}_h)_\Omega,
\end{aligned}
$$

Algebraic system

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v}, \\
\begin{bmatrix} \mathbf{M}_\rho & 0 \\ 0 & \mathbf{M}_C \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \widetilde{\mathbf{v}} \\ \mathbf{s} \end{pmatrix} &= \begin{bmatrix} 0 & -\mathbf{L}^\top(\mathbf{q}) \\ \mathbf{L}(\mathbf{q}) & 0 \end{bmatrix} \begin{pmatrix} \widetilde{\mathbf{v}} \\ \mathbf{s} \end{pmatrix}.
\end{aligned}
$$

The algebraic system can be written more compactly in the following form

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v}, \\
\mathbf{H}\dot{\mathbf{x}} &= \mathbf{J}(\mathbf{q})\mathbf{x}.
\end{aligned}
$$

Summary

Recall on symplectic integrators

Consider the ODE

$$\dot{\mathbf{q}} = \mathbf{v},$$
$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{f}(\mathbf{q}).$$

Störmer Verlet preserves symmetry by staggering

$$\frac{\mathbf{q}_{n+\frac{1}{2}} - \mathbf{q}_{n-\frac{1}{2}}}{\Delta t} = \mathbf{v}_n,$$
$$\mathbf{M}\frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} = \mathbf{f}(\mathbf{q}_{n+\frac{1}{2}})$$

This leads to angular momentum preservation.
Consider the generic ODE $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$. Implicit midpoint update:

$$\frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\Delta t} = \mathbf{g}\left(\frac{\mathbf{x}_{n+1} + \mathbf{x}_n}{2}\right)$$

If system is linear and Hamiltonian then energy is preserved.

Best of both worlds

Idea Combine Störmer–Verlet (leapfrog) and implicit midpoint.

$$
\begin{aligned}
\frac{\mathbf{q}_{n+\frac{1}{2}} - \mathbf{q}_{n-\frac{1}{2}}}{\Delta t} &= \mathbf{v}_n, \\
\mathbf{H}\frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\Delta t} &= \mathbf{J}(\mathbf{q}_{n+\frac{1}{2}})\frac{\mathbf{x}_{n+1} + \mathbf{x}_n}{2}
\end{aligned}
$$

Features:

- **Energy preservation**, since $H = \frac{1}{2}\mathbf{x}^\top \mathbf{H}\mathbf{x}$ and the implicit midpont preserves quadratic invariants;
- **Angular momentum preservation**. Since the integration of displacement and velocity is performed on a staggered grid;
- **Static condensation of the stress**, since it is discretized by a discontinuous space;
- **Linearly implicit scheme**

Connection with the Scalar Auxiliary Variable (SAV) approach

- Introduced for gradient flows[6], extended to Hamiltonian dynamics[7];
- Efficient and energy-stable scheme for systems with diagonal $\mathbf{M}$.

Separable Hamiltonian system (velocity formulation):

$$\dot{\mathbf{q}} = \mathbf{v}, \quad \mathbf{M}\dot{\mathbf{v}} = -\nabla_{\mathbf{q}} V, \quad V \geq 0$$

Introduce scalar variable ξ:

$$V = \frac{1}{2}\xi^2, \quad \nabla_{\mathbf{q}} V = \xi \nabla_{\mathbf{q}} \xi, \quad \dot{\xi} = (\nabla_{\mathbf{q}} \xi)^\top \mathbf{v}$$

Time integration of Bilbao, Ducceschi, and Zama equivalent to the proposed one:

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{v}, \\ \mathbf{H}\dot{\mathbf{x}} &= \mathbf{J}(\mathbf{q})\mathbf{x}, \qquad \mathbf{x} = [\mathbf{v}^\top \ \xi]^\top, \qquad H = \frac{1}{2}\mathbf{x}^\top \mathbf{H}\mathbf{x} \end{aligned}$$

[6] Shen, Xu, and Yang, "The scalar auxiliary variable (SAV) approach for gradient flows"
[7] Bilbao, Ducceschi, and Zama, "Explicit exactly energy-conserving methods for Hamiltonian systems"

Comparison with the energy-momentum preserving scheme of Simo[8]

The idea applies to generic constitutive laws.
Mechanical system:

$$\dot{\mathbf{q}} = \mathbf{v},$$
$$\mathbf{M}\dot{\mathbf{v}} = -\mathbf{L}^\top(\mathbf{q})\boldsymbol{\sigma}(\mathbf{q})$$

Strain energy (geometric nonlinearity only):

$$V_{\mathrm{def}} = \frac{1}{2}\boldsymbol{\varepsilon}^\top \mathbf{W}\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \mathbf{G}(\mathbf{q}),$$

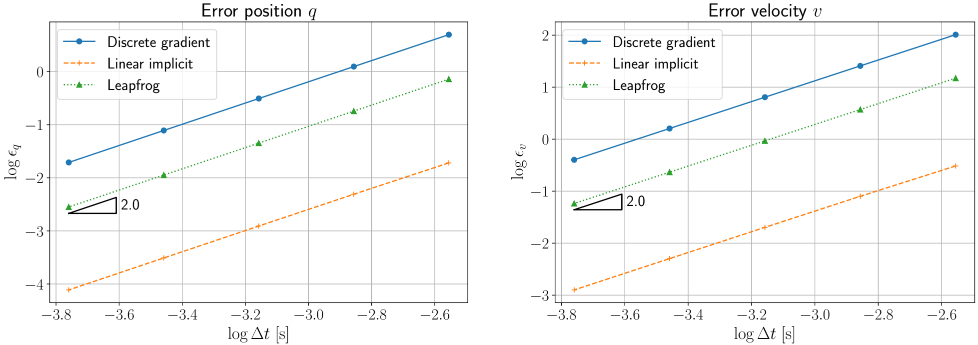
Stress is obtained by derivation $\boldsymbol{\sigma} = \mathbf{W}\mathbf{G}(\mathbf{q})$.
Modified midpoint: use average stress

$$\widehat{\boldsymbol{\sigma}}_{n+1/2} = \mathbf{W}\frac{\mathbf{G}(\mathbf{q}_{n+1}) + \mathbf{G}(\mathbf{q}_n)}{2}$$

[8] Simo and Tarnow, "The discrete energy-momentum method. Conserving algorithms for nonlinear elastodynamics"

Summary

Duffing oscillator: convergence rates



(a) Displacement q (b) Velocity v

Figure: Convergence rate for the Duffing oscillator

Duffing Oscillator: energy preservation and efficiency

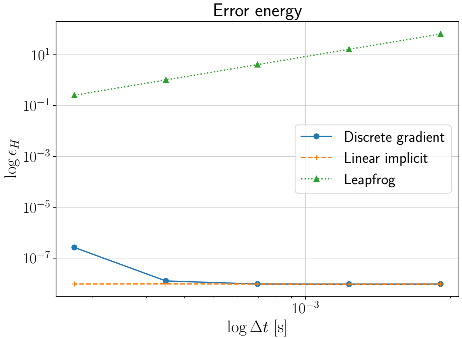


Figure: Energy error for Duffing

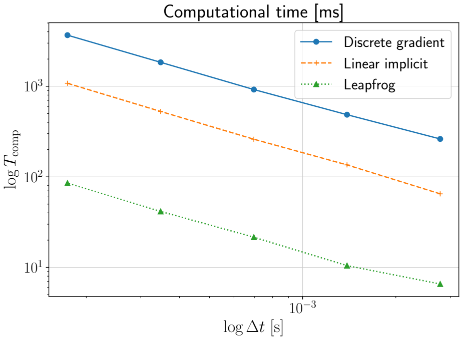
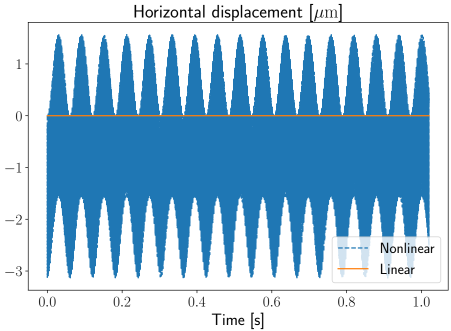
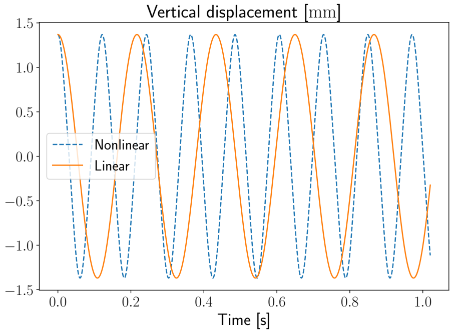


Figure: Computational time for Duffing

Von Kármán beam: vibration staring from first bending mode



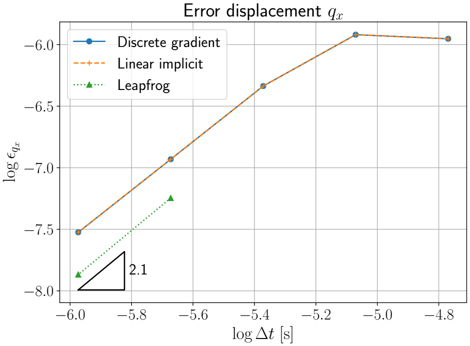
(a) Horizontal displacement $q_x(L/4, t)$



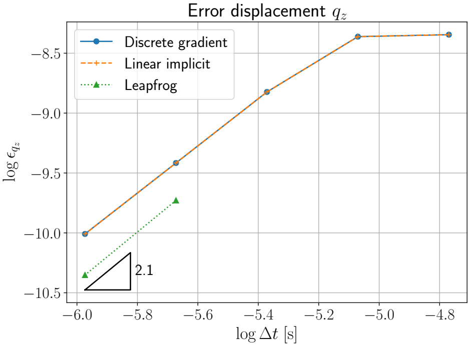
(b) Linear Vertical displacement $q_z(L/4, t)$

Figure: Comparison between linear and nonlinear case at $x = L/4$ for the von-Kármán beam

Von Kármán beam: Convergence rates



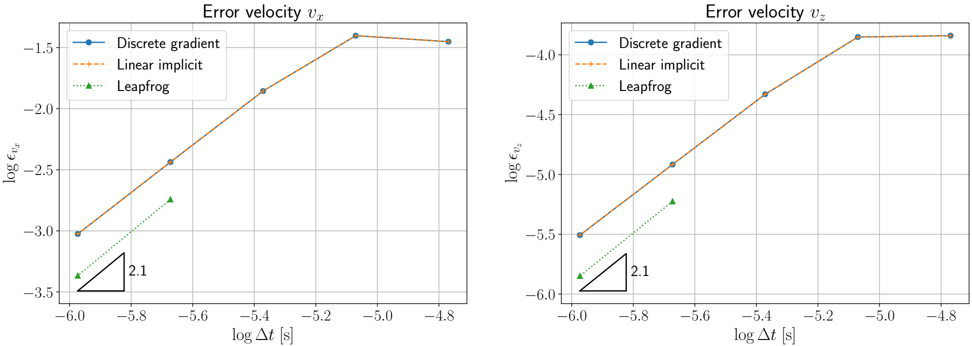
(a) Horizontal displacement q_x



(b) Vertical displacement q_z

Figure: Convergence of q_x, q_z, v_x, v_z over $[0, T_{1,\text{bend}}/10]$

Von Kármán beam: Convergence rates



(a) Horizontal velocity v_x

(b) Vertical velocity v_z

Figure: Convergence of q_x, q_z, v_x, v_z over $[0, T_{1,\mathrm{bend}}/10]$

Von Kármán beam: energy preservation and computational time

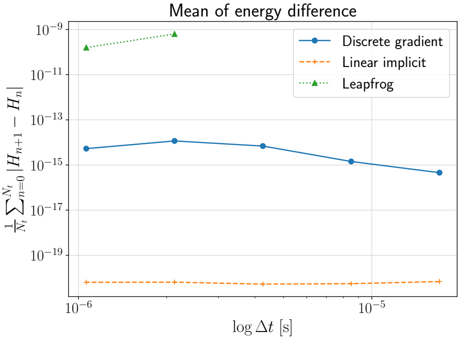


Figure: Mean of $|H_{n+1} - H_n|$ for the von-Kármán beam

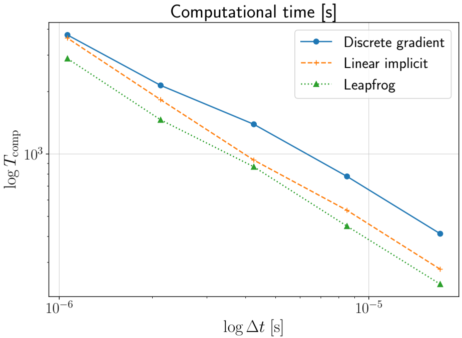
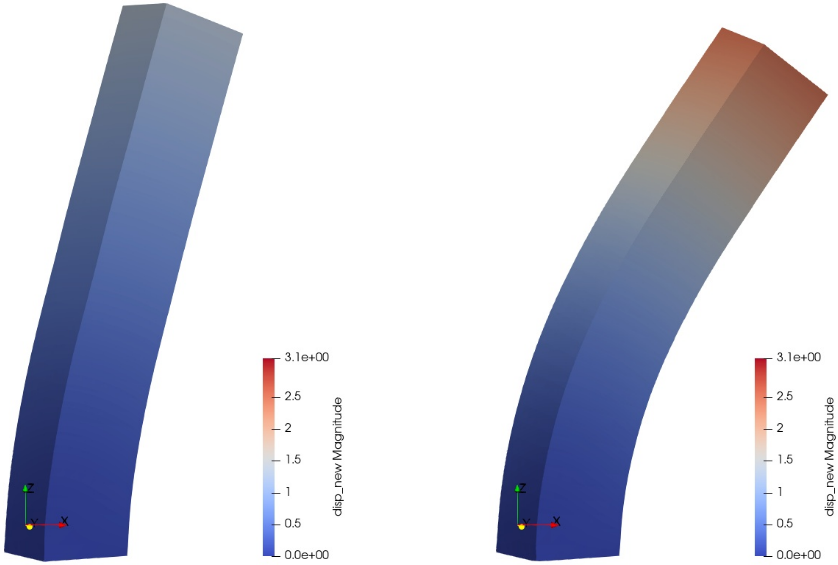
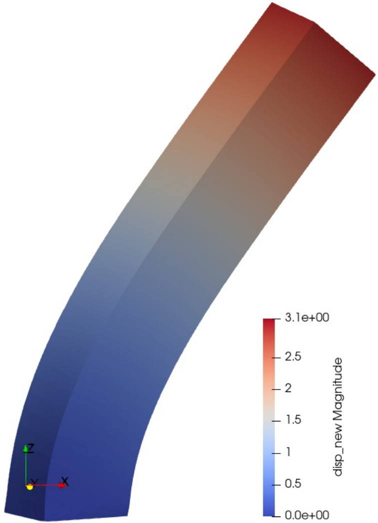


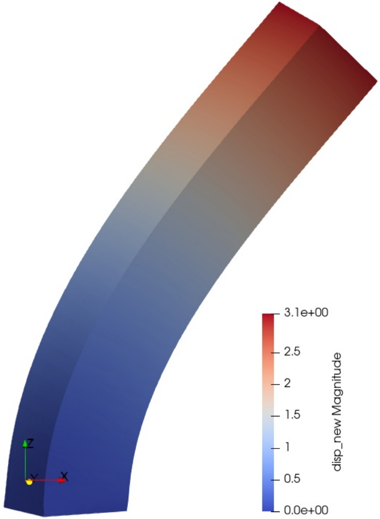
Figure: Computational time for the von-Kármán beam

Nonlinear elasticity: screenshots

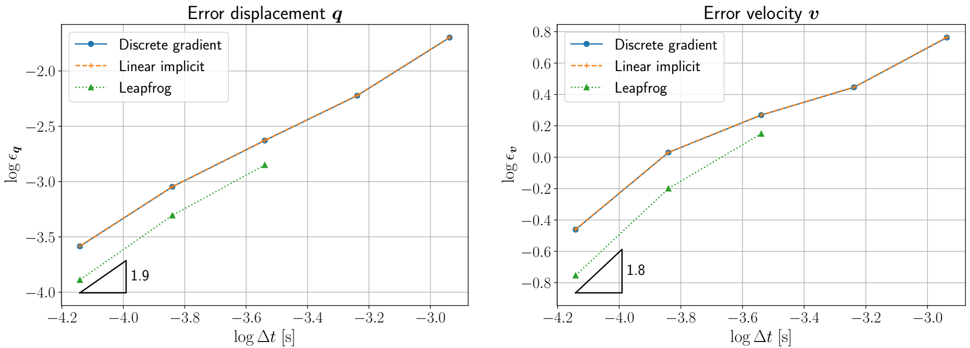


Nonlinear elasticity: screenshots





Nonlinear elasticity: convergence rates



(a) Displacement $\boldsymbol{q}$

(b) Velocity $\boldsymbol{v}$

Convergence rate for $\boldsymbol{q}$, $\boldsymbol{v}$ in geometrically nonlinear elasticity

Nonlinear elasticity: energy preservation and computational time

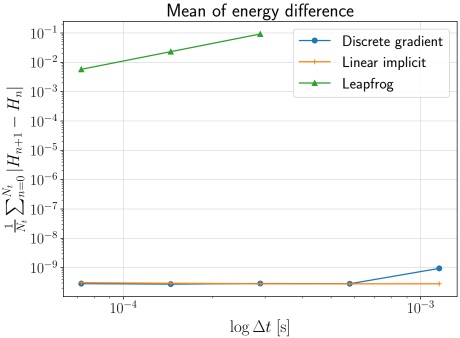


Figure: Mean of $|H_{n+1} - H_n|$ for the geometrically nonlinear elasticity problem

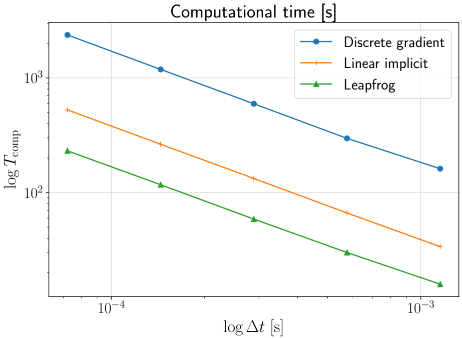


Figure: Computational time for geometrically nonlinear elasticity problem

Conclusions and outlook

Method	Accuracy	Stability	Efficiency
Discrete Gradient	✓	✓	✗
Linear Implicit	✓	✓	✓
Leapfrog	✓	✗	✓

Comparison of time integration methods for geometrically nonlinear mechanics

- General framework: recast geometrically nonlinear problems into a **non-canonical Hamiltonian formulation**
- Extendable to rods, shells, solids
- Mixed FEM with discontinuous stress $\Rightarrow$ static condensation
- Time integration: Störmer–Verlet + implicit midpoint $\Rightarrow$ exact energy-momentum conservation without nonlinear solves
- Outlook: improved linear algebra solvers, FEM–FDM coupling for efficiency

Andrea Brugnoli, Denis Matignon, and Joseph Morlier. "A linearly-implicit energy-momentum preserving scheme for geometrically nonlinear mechanics based on non-canonical Hamiltonian formulations". In: Nonlinear Dynamics (July 2025). ISSN: 1573-269X. DOI: 10.1007/s11071-025-11601-6. URL: https://doi.org/10.1007/s11071-025-11601-6

Different representations leads to qualitative different systems

Geometrically exact beams[9], all variables are represented in the body attached frame (convective representation):

$$\mathrm{Diag}\begin{bmatrix}\rho A\\ \rho \boldsymbol{J}\\ \boldsymbol{C}_t\\ \boldsymbol{C}_r\end{bmatrix}\partial_t\begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{W}\\ \boldsymbol{N}\\ \boldsymbol{M}\end{pmatrix} = +\begin{bmatrix}0 & 0 & \partial_s & 0\\ 0 & 0 & 0 & \partial_s\\ \partial_s & 0 & 0 & 0\\ 0 & \partial_s & 0 & 0\end{bmatrix}\begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{W}\\ \boldsymbol{N}\\ \boldsymbol{M}\end{pmatrix}$$
$$+\begin{bmatrix}0 & [\boldsymbol{\Pi}_V]_\times & [\boldsymbol{K}]_\times & 0\\ [\boldsymbol{\Pi}_V]_\times & [\boldsymbol{\Pi}_W]_\times & [\boldsymbol{\Gamma}+\boldsymbol{e}_1]_\times & [\boldsymbol{K}]_\times\\ [\boldsymbol{K}]_\times & [\boldsymbol{\Gamma}+\boldsymbol{e}_1]_\times & 0 & 0\\ 0 & [\boldsymbol{K}]_\times & 0 & 0\end{bmatrix}\begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{W}\\ \boldsymbol{N}\\ \boldsymbol{M}\end{pmatrix}.$$

No dependence on the displacement in $\mathcal{J}$ but rather on the dual state

$$\mathcal{H}\partial_t\boldsymbol{x} = \mathcal{J}(\boldsymbol{x})\boldsymbol{x}.$$

[9]Hodges, "Geometrically Exact, Intrinsic Theory for Dynamics of Curved and Twisted Anisotropic Beams"

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