An Hamiltonian approach for modular modelling in solid mechanics (but not only)

Andrea Brugnoli

February 9, 2026

Institut Supérieur de l'Aéronautique et de l'Espace
SUPAERO

Summary

How it all started: Modular modeling for control

Port-Hamiltonian systems

Finite elements as interconnections

Applications

- Multibody mechanics
- Thermoelasticity as multiphysical coupling
- Explicit-implicit integration of problems in fluid mechanics

Two words about my academic journey

- 2011-2014 Bachelor in Mechanical Engineering (Politecnico di Milano)

Two words about my academic journey

- 2015-2017: Double degree in space engineering (ISAE - Politecnico di Milano).

Two words about my academic journey

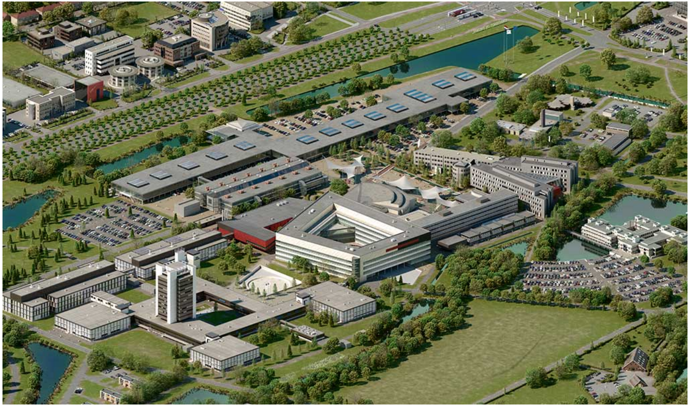
- 2017-2020: PhD in Automatic control (ISAE)

Two words about my academic journey

- 2019: Invited researcher (ITA Brésil).

Two words about my academic journey

- 2020-2022: PostDoc (TU Twente, The Netherlands).



Two words about my academic journey

- 2023: PostDoc (TU Berlin, Germany).

Two words about my academic journey

- 2023 - Assistant Professor at DMSM (ISAE).

Summary

How it all started: Modular modeling for control

Port-Hamiltonian systems

Finite elements as interconnections

Applications

Modeling large flexible space structures

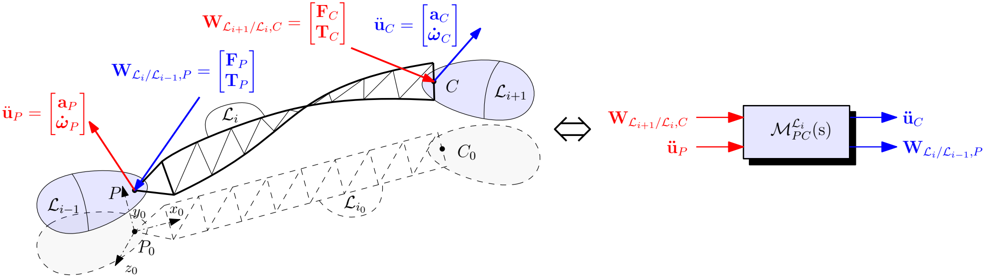
Modular modeling tools are important:

- To simplify validation and verification;
- To speed up prototyping;
- To design robust controllers (*Robust Control Toolbox* from Simulink®, parametric uncertainty, H^∞ optimization, μ analysis).

A theoretical framework for modeling mechanical systems avoiding algebraic constraints xists[1]. It is based on system theory.

[1] D. Alazard and Sanfedino, "A short course on TITOP models for space system modelling".

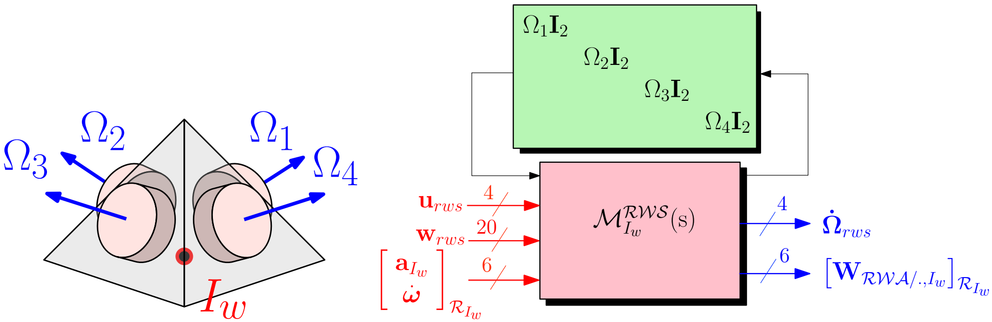
Modular modeling of a satellite[2]



Flexible appendage

[2]Sanfedino et al., "Advances in fine line-of-sight control for large space flexible structures".

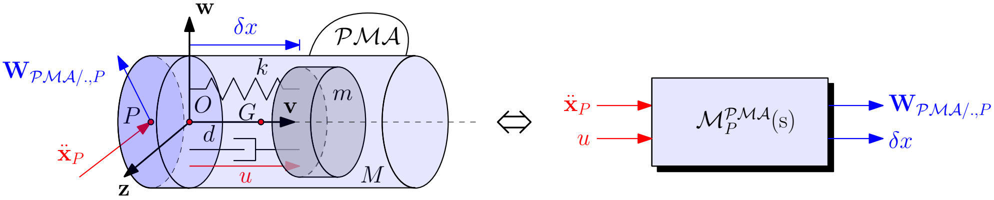
Modular modeling of a satellite[2]



Vibration sources: 4 reaction wheels

[2] Sanfedino et al., "Advances in fine line-of-sight control for large space flexible structures".

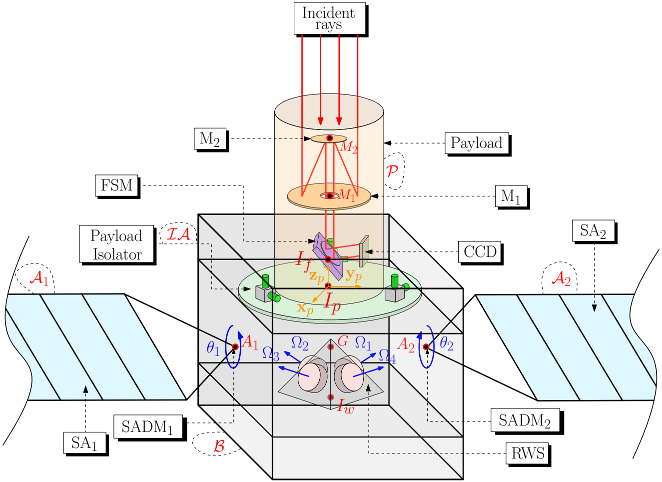
Modular modeling of a satellite[2]



Vibration Mitigation Devices: proof-mass

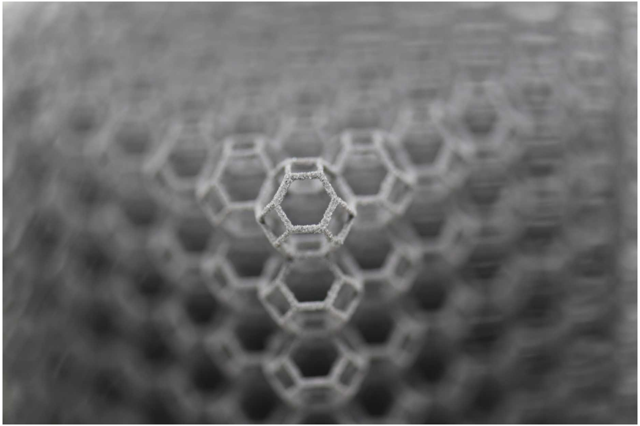
[2] Sanfedino et al., "Advances in fine line-of-sight control for large space flexible structures".

Modular modeling of a satellite[2]



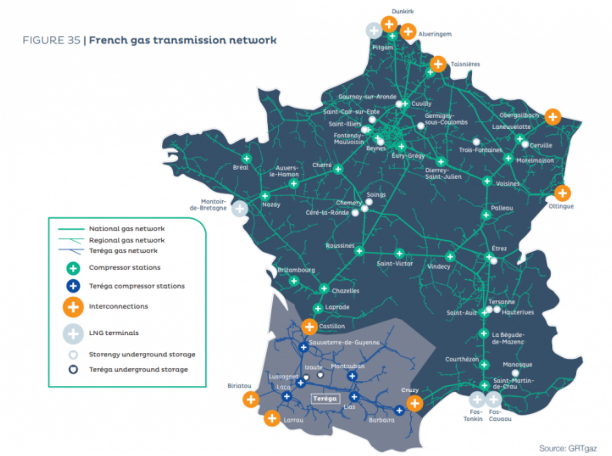
[2]Sanfedino et al., "Advances in fine line-of-sight control for large space flexible structures".

Interconnected systems are everywhere



Metamaterial obtained by assembling lattices into a network.
https://mm.ethz.ch/research-overview/metamaterials/truss-metamaterials.html

Interconnected systems are everywhere



Plan of the French gas network
`https://www.europeangashub.com/report-presentation/french-gas-network-plan`

Summary

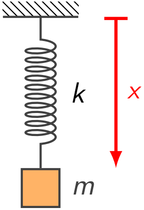
How it all started: Modular modeling for control

Port-Hamiltonian systems

Finite elements as interconnections

Applications

State-space formalism



Example: one-dof oscillator

Newton's law : $m\ddot{x} + kx = f\,.$

Dynamics : $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix},$$

Energy : $$H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

Energy conservation is not necessarily evident from the equations

Hamiltonian formalism

Let's introduce the linear momentum $p := \partial_{\dot{x}} L = m\dot{x}$.
Total energy (Hamiltonian) $H(x,p) = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} \partial_x H \\ \partial_p H \end{pmatrix}.$$

Notice that

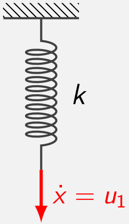
- $\partial_x H = kx$ is the elastic force
- $\partial_p H = \dot{x}$ is the velocity

Can we explain this formalism in a more intuitive way?

The idea of interconnection

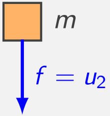
Spring

$$\dot{x} = u_1,$$
$$y_1 = \partial_x U,$$
$$U = \frac{1}{2}kx^2.$$



Mass

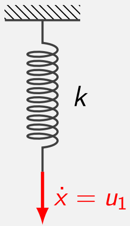
$$\dot{p} = u_2,$$
$$y_2 = \partial_p T,$$
$$T = \frac{1}{2m}p^2.$$



The idea of interconnection

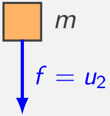
Spring

$$\dot{x} = u_1,$$
$$y_1 = \partial_x U,$$
$$U = \frac{1}{2}kx^2.$$



Mass

$$\dot{p} = u_2,$$
$$y_2 = \partial_p T,$$
$$T = \frac{1}{2m}p^2.$$



Interconnection

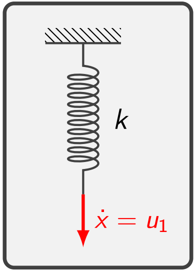
The mass velocity is the spring input.
The elastic force is the mass input.

$$u_1 = y_2, \qquad u_2 = -y_1.$$

The idea of interconnection

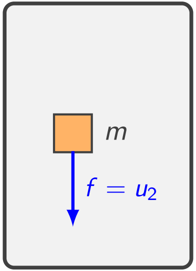
Spring

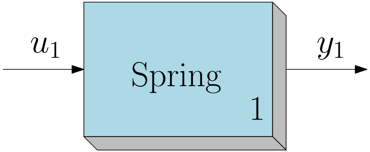
$$\dot{x} = u_1,$$
$$y_1 = \partial_x U,$$
$$U = \frac{1}{2} k x^2.$$

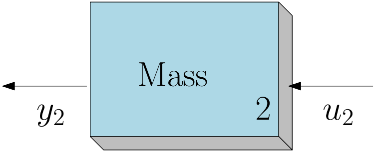


Mass

$$\dot{p} = u_2,$$
$$y_2 = \partial_p T,$$
$$T = \frac{1}{2m} p^2.$$



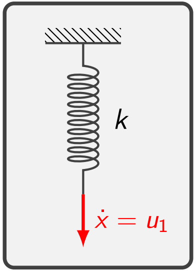




The idea of interconnection

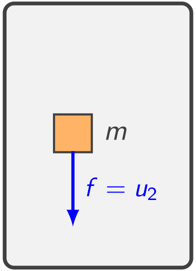
Spring

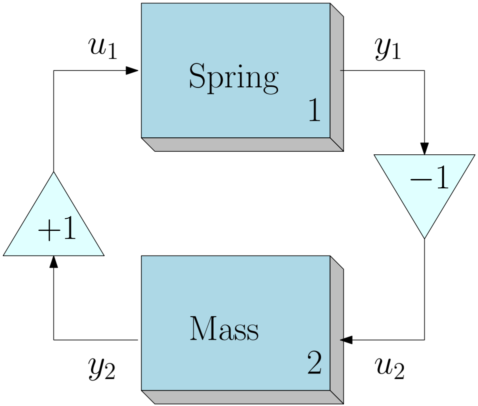
$$\dot{x} = u_1,$$
$$y_1 = \partial_x U,$$
$$U = \frac{1}{2}kx^2.$$



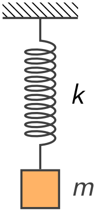
Mass

$$\dot{p} = u_2,$$
$$y_2 = \partial_p T,$$
$$T = \frac{1}{2m}p^2.$$



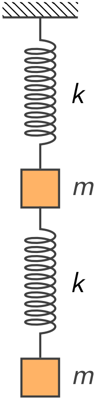


Interconnected mechanical systems[3]



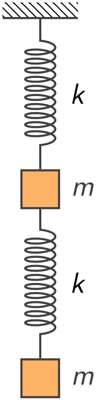
[3] Schaft and Maschke, "Port-Hamiltonian Systems on Graphs".

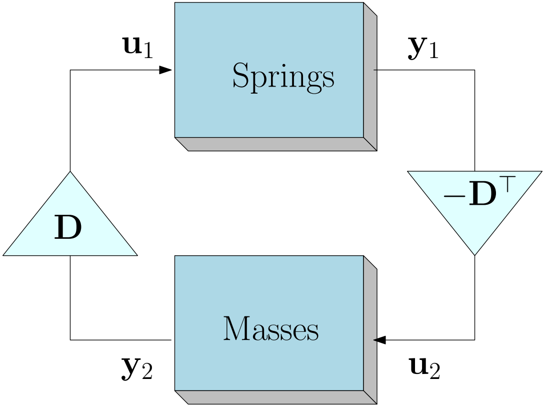
Interconnected mechanical systems[3]



[3] Schaft and Maschke, "Port-Hamiltonian Systems on Graphs".

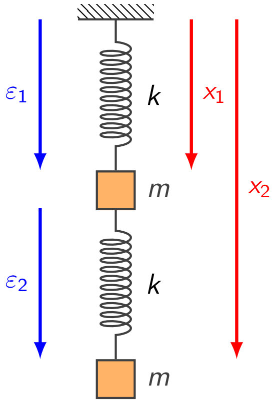
Interconnected mechanical systems[3]





[3] Schaft and Maschke, "Port-Hamiltonian Systems on Graphs".

Interconnected mechanical systems[3]



A **graph** is associated to the system:

- each node corresponds with an inertial elment;
- each edge corresponds to a spring;

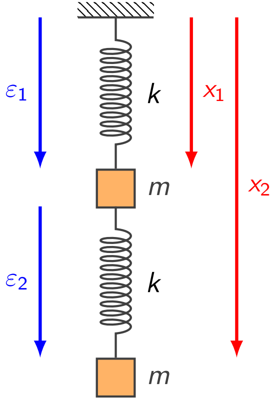
D describes the graph topology.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathbf{p}\end{pmatrix} = \begin{bmatrix}0 & \mathbf{D}\\ -\mathbf{D}^\top & 0\end{bmatrix}\begin{pmatrix}\partial_{\boldsymbol{\varepsilon}} H\\ \partial_{\mathbf{p}} H\end{pmatrix}.$$

- $\boldsymbol{\varepsilon} = (\varepsilon_1 \quad \varepsilon_2)^\top$ spring elongations;
- $\mathbf{p} = (p_1 \quad p_2)^\top$ linear momenta;
- $H = \frac{1}{2}k||\boldsymbol{\varepsilon}||^2 + \frac{1}{2m}||\mathbf{p}||^2$.

[3]Schaft and Maschke, "Port-Hamiltonian Systems on Graphs".

Interconnected mechanical systems[3]



This is different than the canonical Hamiltonian formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} = \begin{bmatrix}0 & \mathbf{I}\\ -\mathbf{I} & 0\end{bmatrix}\begin{pmatrix}\partial_{\mathbf{x}} H\\ \partial_{\mathbf{p}} H\end{pmatrix}.$$

- $\mathbf{x} = (x_1 \quad x_2)^\top$ position of the masses;
- $\mathbf{p} = (p_1 \quad p_2)^\top$ linear momenta;
- $H = \frac{1}{2}k||\mathbf{Dx}||^2 + \frac{1}{2m}||\mathbf{p}||^2$.

[3]Schaft and Maschke, "Port-Hamiltonian Systems on Graphs".

Port-Hamiltonian formalism

Port-Hamiltonian systems :

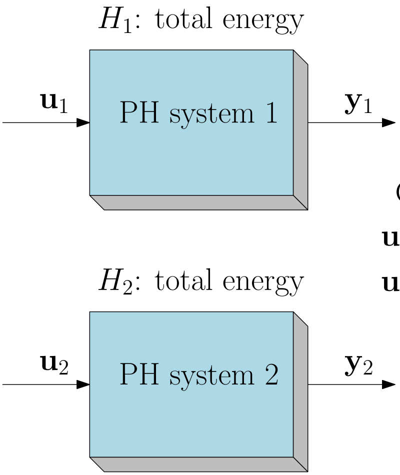
$$\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\nabla H + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{B}^\top \nabla H.$$

- $\mathbf{J} = -\mathbf{J}^\top$ associated to energy conservation;
- $\mathbf{R} \geq 0$ associated to energy dissipation.

Power flow (passivity)

$$\dot{H}(\mathbf{x}) = \nabla H^\top \dot{\mathbf{x}} \leq \mathbf{u}^\top \mathbf{y}.$$

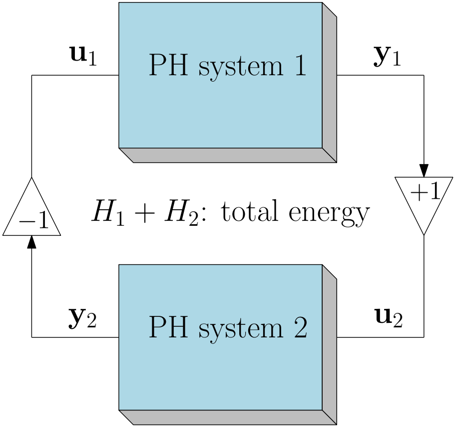
Two energy preserving interconnections



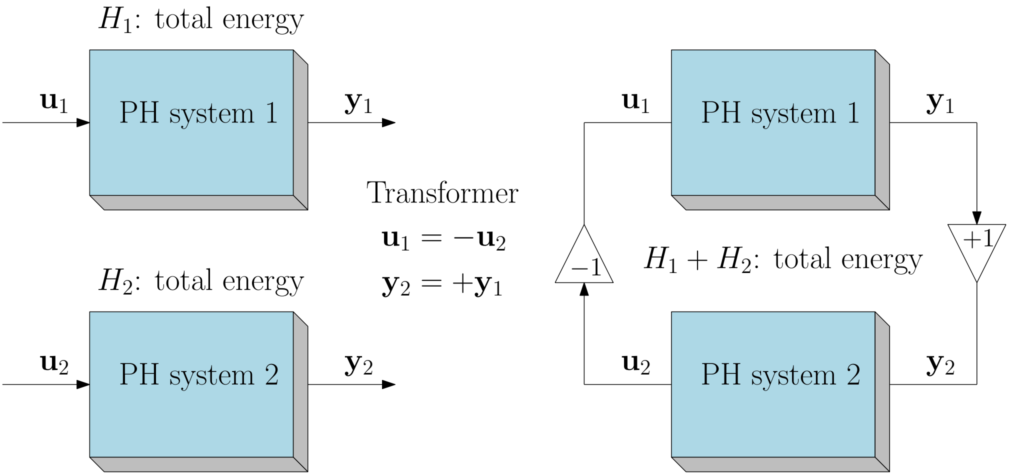
Gyrator

$\mathbf{u}_1 = -\mathbf{y}_2$

$\mathbf{u}_2 = +\mathbf{y}_1$



Two energy preserving interconnections



Multiphysics

This same formalism applies to continuous systems:

- solid mechanics;
- fluid mechanics;
- electromagnetism;
- thermodynamics;

Question: how to discretize the equations?
(Mixed) Finite elements

Summary

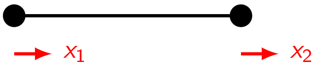
How it all started: Modular modeling for control

Port-Hamiltonian systems

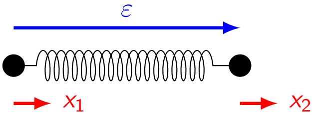
Finite elements as interconnections

Applications

A (mixed) finite element is a port-Hamiltonian system



A (mixed) finite element is a port-Hamiltonian system

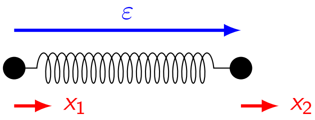


Beam under axial tension:

- Elastic energy: $U = \frac{1}{2} E A \varepsilon^2$;
- Kinetic energy: $T = \frac{1}{2} \mathbf{p}^\top \mathbf{M}^{-1} \mathbf{p}$.

Spring	Mass
$\dot{\varepsilon} = u_1,$	$\dot{\mathbf{p}} = \mathbf{u_2},$
$y_1 = \partial_\varepsilon U.$	$\mathbf{y}_2 = \partial_{\mathbf{p}} T.$

A (mixed) finite element is a port-Hamiltonian system



Beam under axial tension:

- Elastic energy: $U = \frac{1}{2}EA\varepsilon^2$;
- Kinetic energy: $T = \frac{1}{2}\mathbf{p}^\top \mathbf{M}^{-1}\mathbf{p}$.

Spring

$$\dot{\varepsilon} = u_1,$$
$$y_1 = \partial_\varepsilon U.$$

Mass

$$\dot{\mathbf{p}} = \mathbf{u_2},$$
$$\mathbf{y}_2 = \partial_\mathbf{p} T.$$

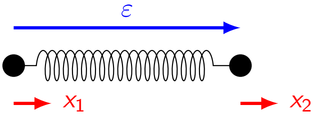
Interconnection

$$u_1 = \begin{bmatrix} -1 & 1 \end{bmatrix} \mathbf{y}_2, \qquad \mathbf{u}_2 = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} y_1.$$

In terms of physical quantities:

$$\dot{\varepsilon} = \begin{bmatrix} -1 & 1 \end{bmatrix} \dot{\mathbf{x}}, \qquad \mathbf{f} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} EA\varepsilon.$$

A (mixed) finite element is a port-Hamiltonian system



Beam under axial tension:

- Elastic energy: $U = \frac{1}{2} EA\varepsilon^2$;
- Kinetic energy: $T = \frac{1}{2}\mathbf{p}^\top \mathbf{M}^{-1}\mathbf{p}$.

Spring

$$\dot{\varepsilon} = u_1, \qquad y_1 = \partial_\varepsilon U.$$

Mass

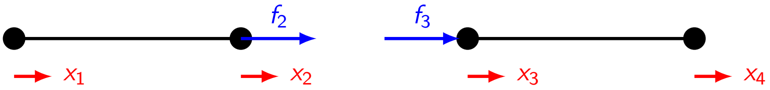
$$\dot{\mathbf{p}} = \mathbf{u}_2, \qquad \mathbf{y}_2 = \partial_{\mathbf{p}} T.$$

The following system is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{p} \\ \varepsilon\end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{d}^\top \\ \mathbf{d} & 0 \end{bmatrix}\begin{pmatrix}\partial_{\mathbf{p}} H \\ \partial_\varepsilon H\end{pmatrix},$$

with $\mathbf{d} = [-1 \quad 1]$.

Finite element assembly as interconnection



Each element is a system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{p}\\ \varepsilon\end{pmatrix} = \begin{bmatrix}0 & -\mathbf{d}^\top\\ \mathbf{d} & 0\end{bmatrix}\begin{pmatrix}\partial_{\mathbf{p}}H\\ \partial_\varepsilon H\end{pmatrix} + \begin{bmatrix}\mathbf{I}\\ 0\end{bmatrix}\mathbf{f}, \qquad \mathbf{I} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

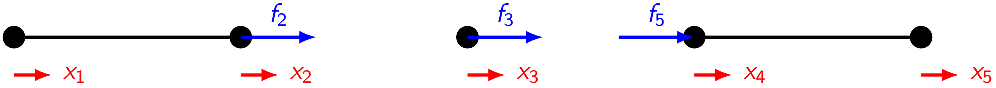
$$\dot{\mathbf{x}} = \begin{bmatrix}\mathbf{I} & 0\end{bmatrix}\begin{pmatrix}\partial_{\mathbf{p}}H\\ \partial_\varepsilon H\end{pmatrix}.$$

The interconnection

$$\dot{x}_2 = \dot{x}_3, \qquad f_2 = -f_3,$$

gives rise to the classical assembly (once Lagrange multipliers are eliminated).

Hybrid methods



More variables (but several computational advantages)[4].
This construction can also be interpreted as an energy preserving interconnection[5].

[4] Park et al., "Displacement-based partitioned equations of motion for structures: Formulation and proof-of-concept applications".
[5] A. Brugnoli, R. Rashad, et al., "Finite element hybridization of port-Hamiltonian systems".

Summary

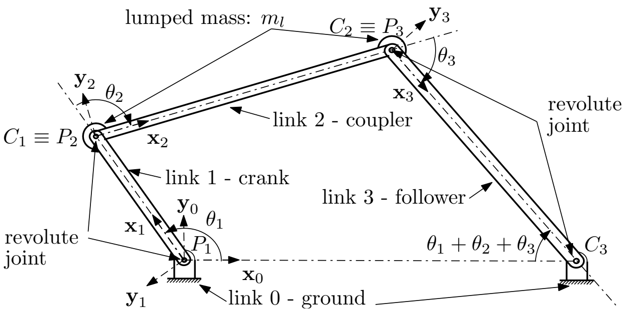
How it all started: Modular modeling for control

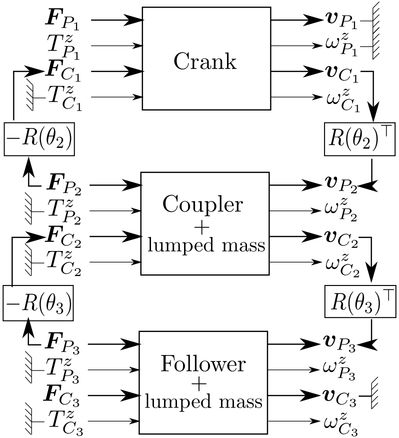
Port-Hamiltonian systems

Finite elements as interconnections

Applications
- Multibody mechanics
- Thermoelasticity as multiphysical coupling
- Explicit-implicit integration of problems in fluid mechanics

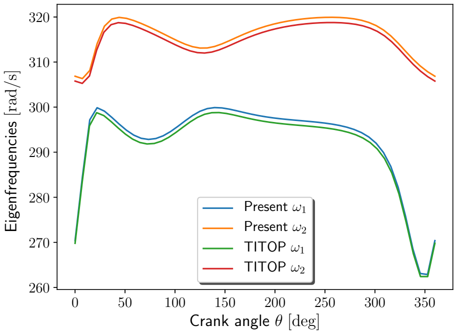
Four bar mechanism[6]

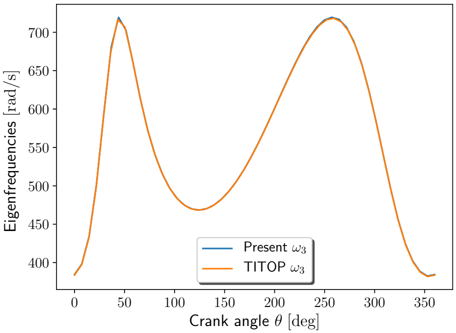




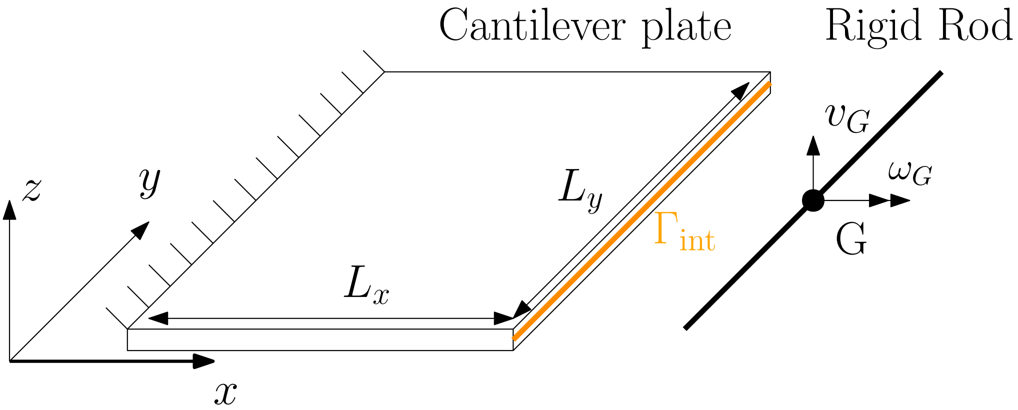
[6] Andrea Brugnoli, Daniel Alazard, et al., "Port-Hamiltonian flexible multibody dynamics".

Eigenproblem

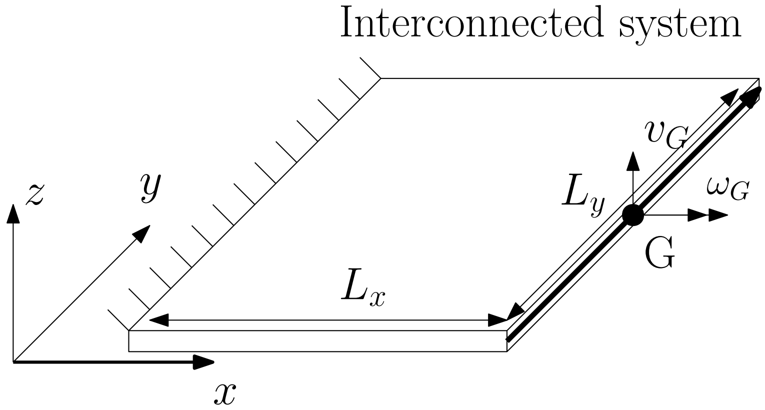




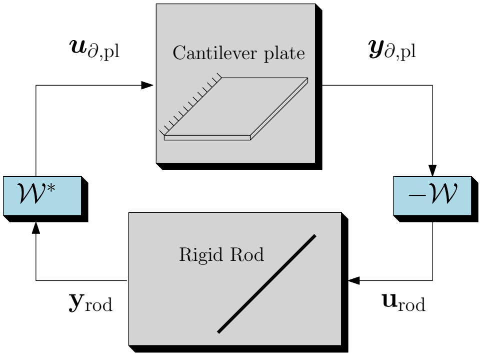
Interconnected Kirchhoff plate

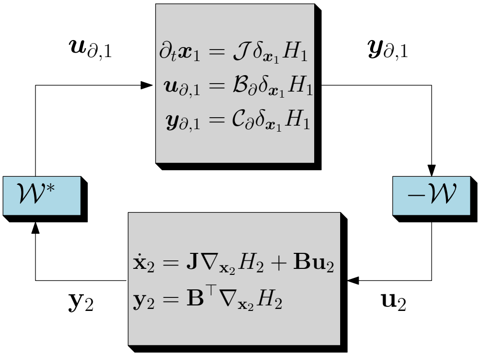


Interconnected Kirchhoff plate



Interconnected Kirchhoff plate





Simulation results

Simulation results

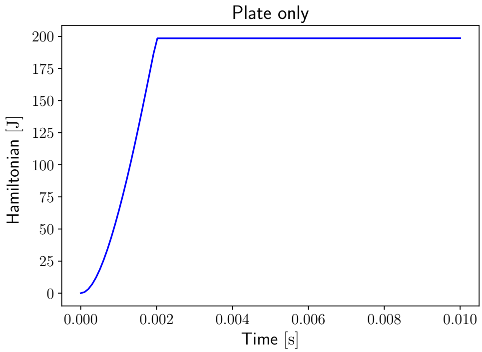


Plate only

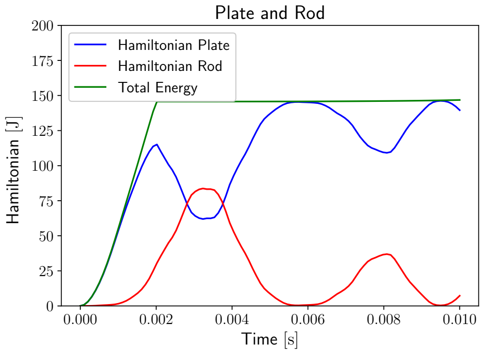


Plate and rod

Summary

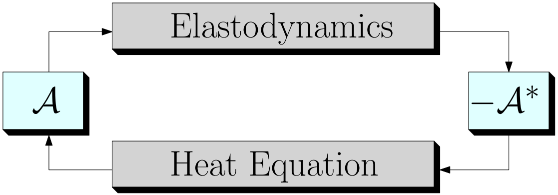
How it all started: Modular modeling for control

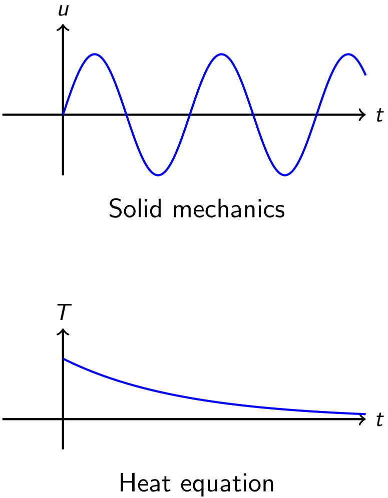
Port-Hamiltonian systems

Finite elements as interconnections

Applications
- Multibody mechanics
- Thermoelasticity as multiphysical coupling
- Explicit-implicit integration of problems in fluid mechanics

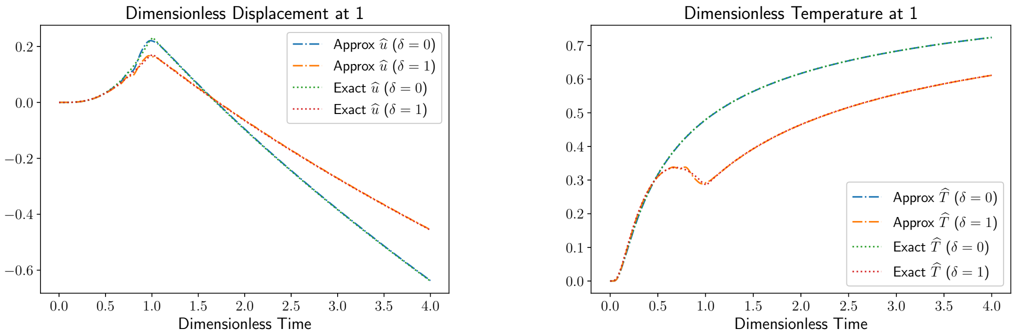
Thermoelasticity as coupled system[7]





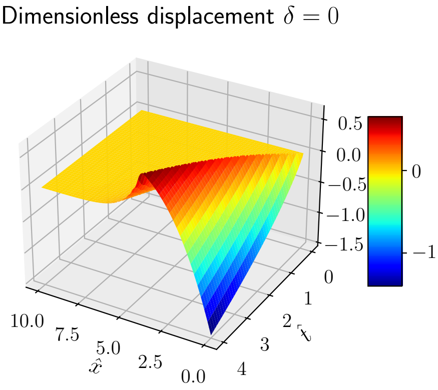
[7] A. Brugnoli, D. Alazard, et al., "A Port-Hamiltonian formulation of linear thermoelasticity and its mixed finite element discretization".

Results for an analytical problem

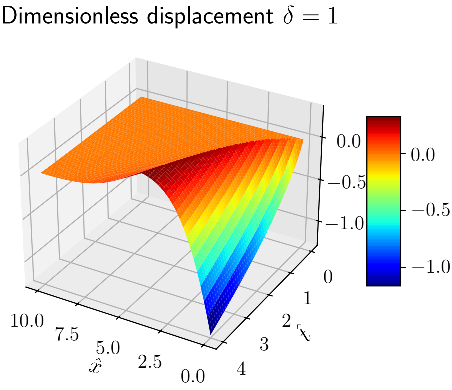


Displacement and temperature at location $\widehat{x} = 1$.

Results for an analytical problem

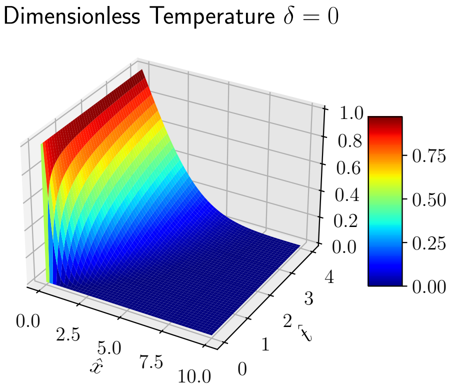


(a) $\delta = 0$

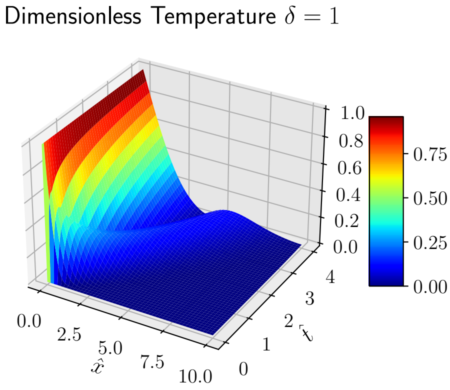


(b) $\delta = 1$

Results for an analytical problem



(a) $\delta = 0$



(b) $\delta = 1$

Summary

How it all started: Modular modeling for control

Port-Hamiltonian systems

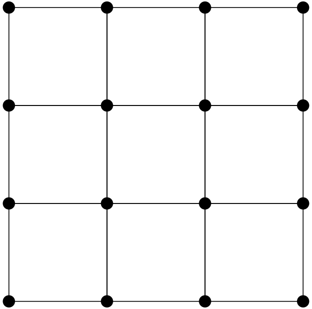
Finite elements as interconnections

Applications

- Multibody mechanics
- Thermoelasticity as multiphysical coupling
- Explicit-implicit integration of problems in fluid mechanics

The dual structure of physics

Physics equations can be written in a primal-dual way[8].

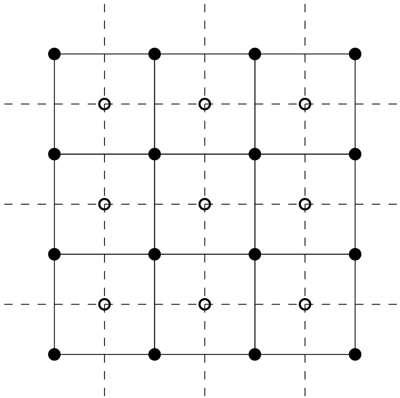


Mesh

[8]Andrea Brugnoli, Ramy Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus".

The dual structure of physics

Physics equations can be written in a primal-dual way[8].

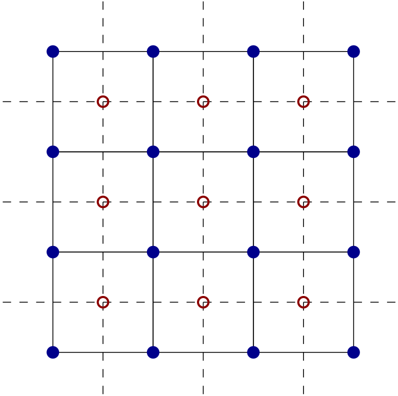


Topological dual mesh

[8] Andrea Brugnoli, Ramy Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus".

The dual structure of physics

Physics equations can be written in a primal-dual way[8].

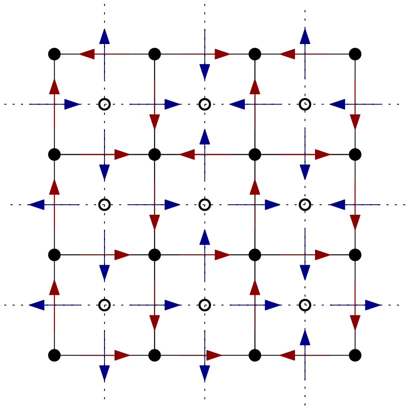


Nodes and their dual

[8] Andrea Brugnoli, Ramy Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus".

The dual structure of physics

Physics equations can be written in a primal-dual way[8].



Edges and their dual

[8] Andrea Brugnoli, Ramy Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus".

The dual structure of physics

Physics equations can be written in a primal-dual way[8].

- pressure and velocity in acoustics;
- electric and magnetic fields in electromagnetism;

[8] Andrea Brugnoli, Ramy Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus".

Euler equations

Equations describing the motion of an ideal fluid

$$\begin{aligned} \partial_t \boldsymbol{u} &= -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p, \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned}$$

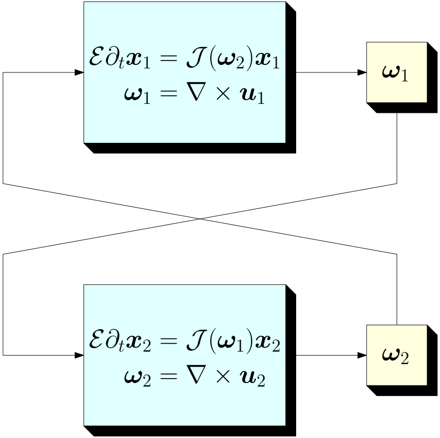
We can rewrite them in rotational form

$$\begin{aligned} \partial_t \boldsymbol{u} &= -\boldsymbol{\omega} \times \boldsymbol{u} - \nabla P, & \boldsymbol{\omega} &:= \nabla \times \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} &= 0. & P &:= p + 1/2\, ||\boldsymbol{u}||^2. \end{aligned}$$

Invariants:

- Energy: $K = \frac{1}{2} \int_\Omega ||\boldsymbol{u}||^2$;
- Helicity: $H = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\omega}$.

Conservative discretization via vorticity exchange[9]



System 1: primal discretization.
System 2: dual discretization.

Explicit-implicit time integration:
only linear system solve.

The scheme conserves

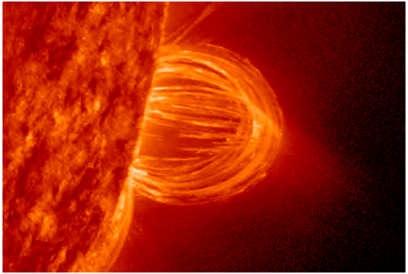
- ✓ Energy;
- ✓ Helicity;

[9] Zhang et al., “A mass-, kinetic energy- and helicity-conserving mimetic dual-field discretization for three-dimensional incompressible Navier-Stokes equations, part I: Periodic domains”.

Shear layer roll-up

Magnetohydrodynamics (MHD)

MHD describes the macroscopic behavior of conductive fluids (plasmas).



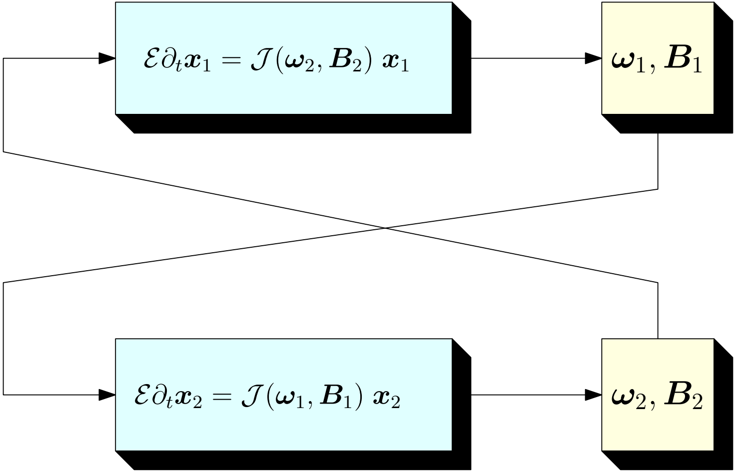
Coupling of:

- Maxwell equations;
- fluid dynamics;

Several invariants ($\boldsymbol{B}$ magnetic field):

- Energy $K = \frac{1}{2}\int_\Omega ||\boldsymbol{u}||^2 + \kappa||\boldsymbol{B}||^2$;
- Cross Helicity $H_c = \int_\Omega \boldsymbol{B}\cdot\boldsymbol{u}$;
- Magnetic Helcity $H_m = \int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}$ where $\nabla\times\boldsymbol{A} = \boldsymbol{B}$.

Conservative discretization by exchange of vorticity and magnetic fields



Orszag-Tang

Bibliography I

- Alazard, D. and F. Sanfedino. "A short course on TITOP models for space system modelling". In: *IFAC-PapersOnLine* 54.12 (2021). IFAC Workshop on Aerospace Control Education WACE 2021, pp. 7–13. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2021.11.002.
- Brugnoli, A., D. Alazard, et al. "A Port-Hamiltonian formulation of linear thermoelasticity and its mixed finite element discretization". In: *Journal of Thermal Stresses* 44.6 (2021), pp. 643–661. DOI: 10.1080/01495739.2021.1917322.
- Brugnoli, A., R. Rashad, et al. "Finite element hybridization of port-Hamiltonian systems". In: (2023).
- Brugnoli, Andrea, Daniel Alazard, et al. "Port-Hamiltonian flexible multibody dynamics". In: *Multibody System Dynamics* 51.3 (Mar. 2021), pp. 343–375. ISSN: 1573-272X. DOI: 10.1007/s11044-020-09758-6.

Bibliography II

- Brugnoli, Andrea, Ramy Rashad, and Stefano Stramigioli. "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus". In: *Journal of Computational Physics* 471 (2022). ISSN: 0021-9991. DOI: 10.1016/j.jcp.2022.111601.
- Park, K. C. et al. "Displacement-based partitioned equations of motion for structures: Formulation and proof-of-concept applications". In: *International Journal for Numerical Methods in Engineering* 124.22 (2023), pp. 5020–5046. DOI: 10.1002/nme.7334.
- Sanfedino, F. et al. "Advances in fine line-of-sight control for large space flexible structures". In: *Aerospace Science and Technology* 130 (2022), p. 107961. ISSN: 1270-9638. DOI: 10.1016/j.ast.2022.107961.
- Schaft, A. J. van der and B. M. Maschke. "Port-Hamiltonian Systems on Graphs". In: *SIAM Journal on Control and Optimization* 51.2 (2013), pp. 906–937. DOI: 10.1137/110840091.

Bibliography III

Zhang, Yi et al. "A mass-, kinetic energy- and helicity-conserving mimetic dual-field discretization for three-dimensional incompressible Navier-Stokes equations, part I: Periodic domains". In: *Journal of Computational Physics* (2021), p. 110868. ISSN: 0021-9991. DOI: 10.1016/j.jcp.2021.110868.