A domain decomposition strategy for natural imposition of mixed boundary conditions

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Summary

Mixed boundary conditions in standard and mixed finite element scheme

Domain decomposition and Interconnection

- The 1D case
- The $\mathbb{R}^d$ case

Summary

Mixed boundary conditions in standard and mixed finite element scheme

Domain decomposition and Interconnection

How to impose essential boundary conditions

Using a standard formulation, the Neumann bc is the natural one, whereas the Dirichlet bc is essential

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{B}\mathbf{f},$$
$$\mathbf{Tr}_{\Gamma_D}\mathbf{q} = \widehat{\mathbf{q}}_D.$$

This is a very common situation in **multibody dynamics**.

The Dirichlet condition can be enforced via:

- Lagrange multiplier;
- elimination of rows;
- penalty (additional parameters).

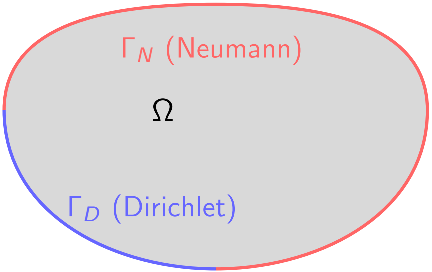

Illustration of the idea

Consider the propagation of longitudinal wave in a rod (unitary cross section)

$$\rho \partial_{tt} q - \partial_x (E \partial_x q) = 0, \qquad + \text{ Boundary conditions.}$$

q is the longitudinal displacement, ρ is the density and E the Young modulus.

This is a Lagrangian formulation, but other equivalent formulation may be used:

- Hamiltonian;
- port-Hamiltonian (or mixed formulation in FEM community).

The port-Hamiltonian formulation for longitudinal waves[1]

The energy doesn't depend on q but only on its time and space derivative

$$H(p, \varepsilon) = \frac{1}{2}\int_0^L \frac{p^2}{\rho} + E\varepsilon^2 \mathrm{d}x, \qquad p := \rho\partial_t q, \quad \varepsilon = \partial_x q.$$

What if we write the equations using the variables that explicitly appear in the energy?

[1] van der Schaft and Maschke, "Hamiltonian formulation of distributed-parameter systems with boundary energy flow".

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What if we write the equations using the variables that explicitly appear in the energy?

Port-Hamiltonian formulation

Two coupled conservation laws are obtained

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon \\ p\end{pmatrix} = \begin{bmatrix}0 & \partial_x \\ \partial_x & 0\end{bmatrix}\begin{pmatrix}\delta_\varepsilon H \\ \delta_p H\end{pmatrix}, \qquad \begin{pmatrix}\delta_\varepsilon H \\ \delta_p H\end{pmatrix} := \begin{bmatrix}E & 0 \\ 0 & \rho^{-1}\end{bmatrix}\begin{pmatrix}\varepsilon \\ p\end{pmatrix}, \quad \begin{matrix}\text{Stress } \sigma, \\ \text{Velocity } v.\end{matrix}$$

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The port-Hamiltonian formulation for longitudinal waves[1]

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What if we write the equations using the variables that explicitly appear in the energy?

Port-Hamiltonian formulation

The system can be written using stress and velocity only

$$\begin{bmatrix} c & 0 \\ 0 & \rho \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix} \begin{pmatrix} \sigma \\ v \end{pmatrix}, \qquad c := \frac{1}{E} \text{ is the compliance.}$$

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Power balance across the boundary and causalities

Power balance: $\dot{H} = v(L)\sigma(L) - v(0)\sigma(0) = v\ \sigma \cdot n\big|_{\partial[0,L]}$.

Possible causalities

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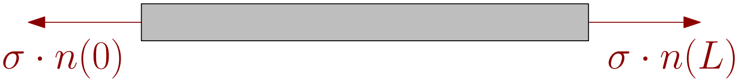
Possible causalities

Free-free (Neumann):

- Input given by the Neumann condition $\mathbf{u}_N = \sigma \cdot n|_{\partial[0,L]}$
- Output given by the Dirichlet condition $\mathbf{y}_D = v|_{\partial[0,L]}$



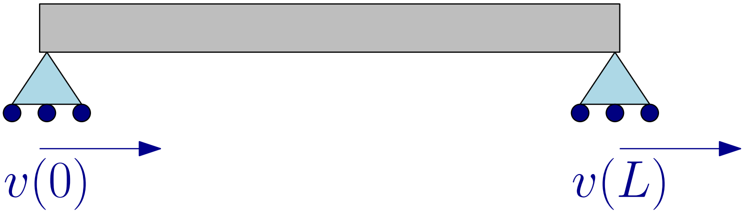
Power balance across the boundary and causalities

Power balance: $\dot{H} = v(L)\sigma(L) - v(0)\sigma(0) = v\ \sigma \cdot n\big|_{\partial[0,L]}.$

Possible causalities

Clamped-clamped (Dirichlet):

- Input given by the Dirichlet condition $\mathbf{u}_D = v|_{\partial[0,L]}$
- Output given by the Neumann condition $\mathbf{y}_N = \sigma \cdot n|_{\partial[0,L]}$



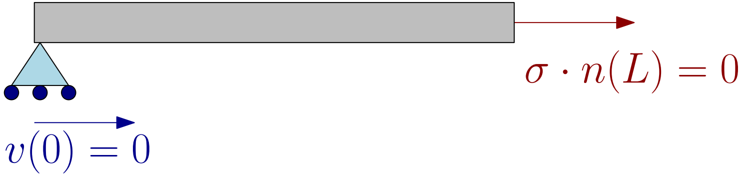
Power balance across the boundary and causalities

Power balance: $\dot{H} = v(L)\sigma(L) - v(0)\sigma(0) = v\ \sigma \cdot n\big|_{\partial[0,L]}$.

Possible causalities

Cantilever (mixed) $\partial[0,L] = \Gamma_D \cup \Gamma_N$

$$
\begin{aligned}
u_N &= \sigma \cdot n\big|_{\Gamma_N}, \qquad & y_D &= v\big|_{\Gamma_N}, \\
u_D &= v\big|_{\Gamma_D}, \qquad & y_N &= \sigma \cdot n\big|_{\Gamma_D}.
\end{aligned}
$$



Discretization via mixed finite elements: the primal dual structure

The discretization proceeds in three steps:

- take the weak formulation;
- perform integration by parts (depending on the causality);
- project on a finite element basis.

Discretization via mixed finite elements: the primal dual structure

First weak formulation: Neumann natural control

Find $\sigma \in L^2(\Omega),\ v \in H^1(\Omega)$

$$
\begin{aligned}
(\xi_\sigma,\, c\,\partial_t \sigma)_\Omega &= +(\xi_\sigma,\, \partial_x v)_\Omega, && \forall\, \xi_\sigma \in L^2(\Omega),\\
(\xi_v,\, \rho\,\partial_t v)_\Omega &= -(\partial_x \xi_v,\, \sigma)_\Omega + (\xi_v,\, \mathbf{u}_N)_{\partial\Omega}, && \forall\, \xi_v \in H^1(\Omega).
\end{aligned}
$$

Discretization via mixed finite elements: the primal dual structure

Second weak formulation: Dirichlet natural control

Find $\sigma \in H^1(\Omega),\ v \in L^2(\Omega)$

$$
\begin{aligned}
(\xi_\sigma,\, c\,\partial_t\sigma)_\Omega &= -(\partial_x\xi_\sigma,\, \sigma)_\Omega + (\xi_\sigma,\, \mathbf{u}_D)_{\partial\Omega}, \qquad && \forall\, \xi_\sigma \in H^1(\Omega),\\
(\xi_v,\, \rho\,\partial_t v)_\Omega &= +(\xi_v,\, \partial_x\sigma)_\Omega, && \forall\, \xi_v \in L^2(\Omega).
\end{aligned}
$$

Finite element basis

The basis for the two variables need to be different to avoid spurious mode

$$\sigma_h(x,t) = \sum_{i=1}^{N_\sigma} \varphi_\sigma^i(x)\sigma_i(t), \qquad v_h(x,t) = \sum_{i=1}^{N_v} \varphi_v^i(x) v_i(t)$$

The bases functions span the corresponding finite element space

$$\sigma_h \in \mathcal{S}_h = \mathrm{span}\{\varphi_\sigma^1, \ldots, \varphi_\sigma^{N_\sigma}\},$$
$$v_h \in \mathcal{V}_h = \mathrm{span}\{\varphi_v^1, \ldots, \varphi_v^{N_v}\},$$

Choice of the finite element basis (Neumann control)

In this formulation

- $v_h \in \mathbb{L}_1 \subset H^1(\Omega)$. **Lagrange elements** (just like in the static case) can be used.
- $\sigma_h \in L^2(\Omega)$. Which finite element space to choose?

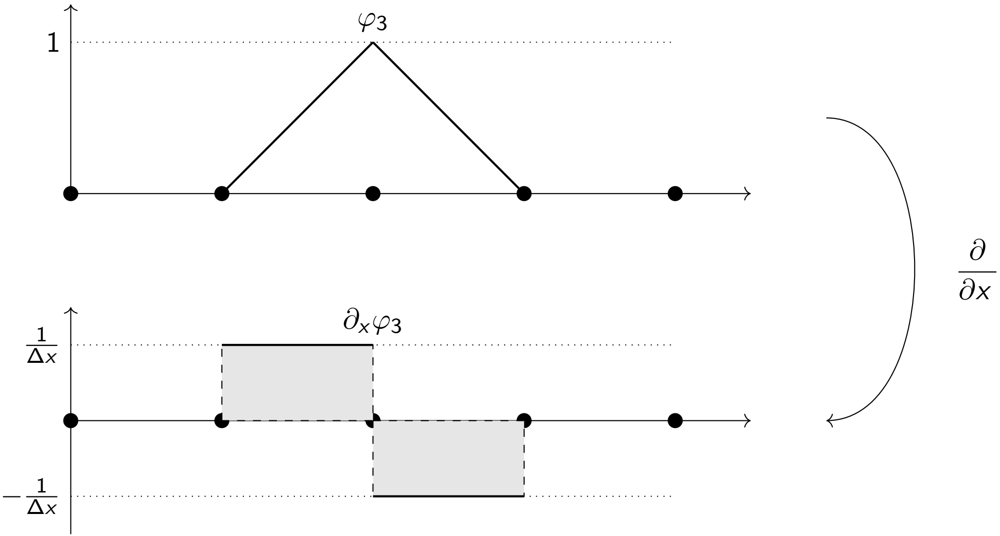
Remind the second equation reads

$$(\xi_\sigma,\, c\,\partial_t \sigma_h)_\Omega = (\xi_\sigma,\, \partial_x v_h)_\Omega.$$

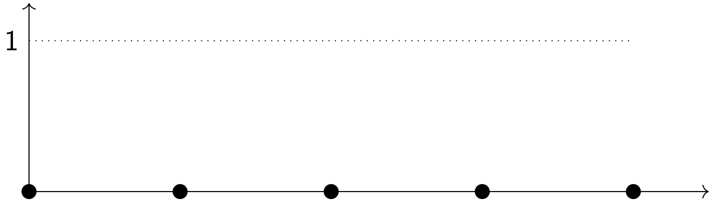
For this equation to hold pointwise, the finite element space should satisfy

$$\partial_x \mathcal{V}_h \subset \mathcal{S}_h, \implies c\,\partial_t \sigma_h = \partial_x v_h, \qquad \text{(if } c \text{ is smooth)}.$$

The derivative of a Lagrange space



The Discontinous Galerkin space $\mathbb{DG}_0$



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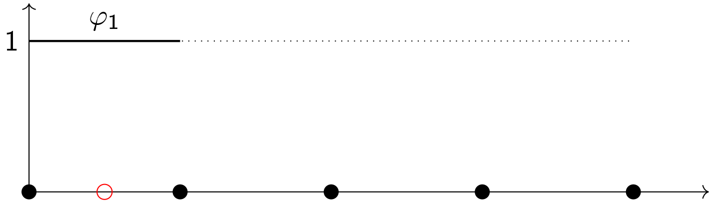

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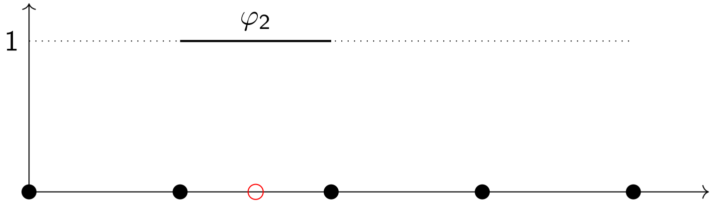

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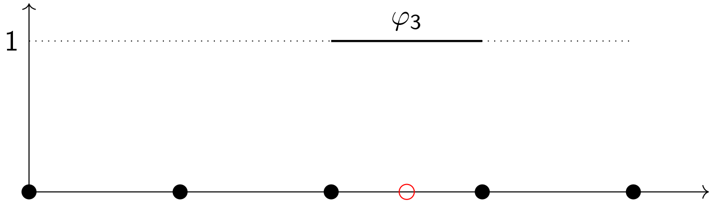

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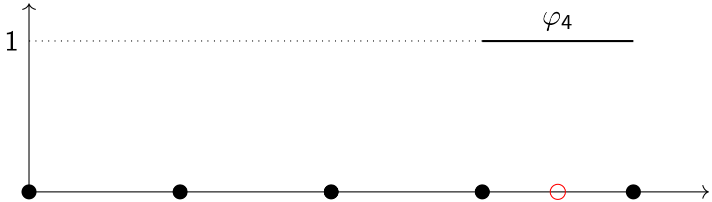

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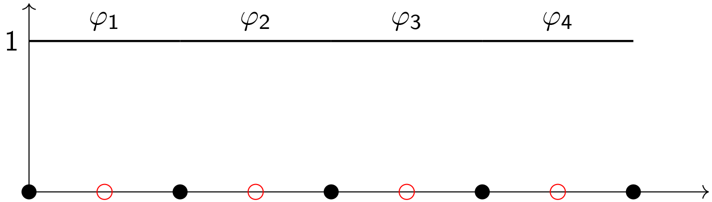

It holds $\partial_x \mathbb{L}_1 \subset \mathbb{DG}_0$. This choice guarantees stability of the formulation.

This is a particular instance of a much more general mathematical construction (subcomplex of an Hilbert complex).

Algebraic system: dynamics

Formulation with Neumann natural control

$$
\begin{bmatrix} \mathbf{M}_c^\sigma & 0 \\ 0 & \mathbf{M}_\rho^v \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{D} \\ -\mathbf{D}^\top & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} + \begin{bmatrix} 0 \\ \mathbf{Tr}^\top \end{bmatrix} \mathbf{u}_N,
$$

$$
\mathbf{y}_D = \begin{bmatrix} 0 & \mathbf{Tr} \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix}.
$$

The matrices are computed as follows

$$
[\mathbf{M}_\rho^v]_{ij} = \int_0^L \rho \varphi_v^i \cdot \varphi_v^j \, \mathrm{d}x, \qquad [\mathbf{M}_c^\sigma]_{ij} = \int_0^L c \varphi_\sigma^i \cdot \varphi_\sigma^j \, \mathrm{d}x, \qquad [\mathbf{D}]_{ij} = \int_0^L \varphi_\sigma^i \cdot \frac{\partial \varphi_v^j}{\partial x} \, \mathrm{d}x.
$$

Tr is a trace matrix

$$
\mathbf{Tr} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.
$$

The dual formulation

For the 1D wave equation, the dual formulation is completely symmetrical.

Formulation with Dirichlet natural control

$$\begin{bmatrix} \mathbf{M}_c^\sigma & 0 \\ 0 & \mathbf{M}_\rho^v \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{D}^\top \\ \mathbf{D} & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix} + \begin{bmatrix} \mathbf{Tr}_n^\top \\ 0 \end{bmatrix} \mathbf{u}_D,$$

$$\mathbf{y}_N = \begin{bmatrix} \mathbf{Tr}_n & 0 \end{bmatrix} \begin{pmatrix} \mathbf{s} \\ \mathbf{v} \end{pmatrix}.$$

$\mathbf{Tr}_n$ is the normal trace matrix

$$\mathbf{Tr}_n = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

Mixed boundary conditions

Partition of the boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ (in 1D each subpartition is 1 point).

$$u_N = \sigma \cdot n\big|_{\Gamma_N}, \qquad u_D = v\big|_{\Gamma_D}.$$

Then the resulting system is a DAE (differential algebraic equation).

Mixed boundary conditions

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Then the resulting system is a DAE (differential algebraic equation).

Primal formulation (mixed control)

$$\mathrm{Diag}\begin{bmatrix}\mathbf{M}_c^\sigma \\ \mathbf{M}_\rho^v \\ 0\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \lambda_N\end{pmatrix} = \begin{bmatrix}0 & \mathbf{D} & 0 \\ -\mathbf{D}^\top & 0 & \mathbf{Tr}_{\Gamma_D}^\top \\ 0 & -\mathbf{Tr}_{\Gamma_D} & 0\end{bmatrix}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \lambda_N\end{pmatrix} + \begin{bmatrix}0 & 0 \\ \mathbf{Tr}_{\Gamma_N}^\top & 0 \\ 0 & 1\end{bmatrix}\begin{pmatrix}u_N \\ u_D\end{pmatrix},$$

$$\begin{pmatrix}y_D \\ y_N\end{pmatrix} = \begin{bmatrix}0 & \mathbf{Tr}_{\Gamma_N} & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}\mathbf{s} \\ \mathbf{v} \\ \lambda_N\end{pmatrix}.$$

Mixed boundary conditions

Partition of the boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ (in 1D each subpartition is 1 point).

$$u_N = \sigma \cdot n|_{\Gamma_N}, \qquad u_D = v|_{\Gamma_D}.$$

Then the resulting system is a DAE (differential algebraic equation).

Dual formulation (mixed control)

$$\mathrm{Diag}\begin{bmatrix} \mathbf{M}_c^\sigma \\ \mathbf{M}_\rho^v \\ 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{s} \\ \mathbf{v} \\ \lambda_D \end{pmatrix} = \begin{bmatrix} 0 & -\mathbf{D}^\top & \mathbf{Tr}_{n,\Gamma_N}^\top \\ \mathbf{D} & 0 & 0 \\ -\mathbf{Tr}_{n,\Gamma_N} & 0 & 0 \end{bmatrix}\begin{pmatrix} \mathbf{s} \\ \mathbf{v} \\ \lambda_D \end{pmatrix} + \begin{bmatrix} \mathbf{Tr}_{n,\Gamma_D}^\top & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} u_D \\ u_N \end{pmatrix},$$

$$\begin{pmatrix} y_N \\ y_D \end{pmatrix} = \begin{bmatrix} \mathbf{Tr}_{n,\Gamma_D} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} \mathbf{s} \\ \mathbf{v} \\ \lambda_D \end{pmatrix}.$$

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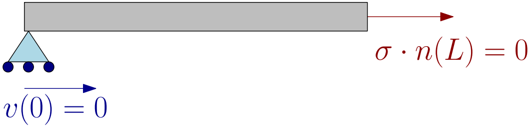
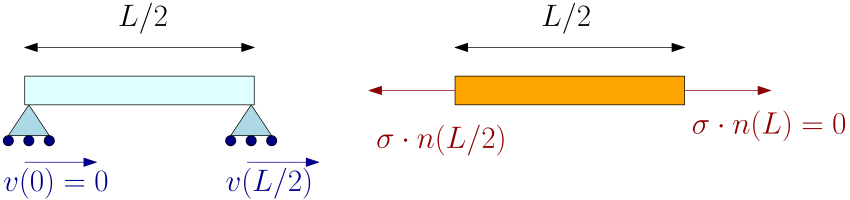
Mixed boundary conditions via interconnection

Consider again the cantilever bar

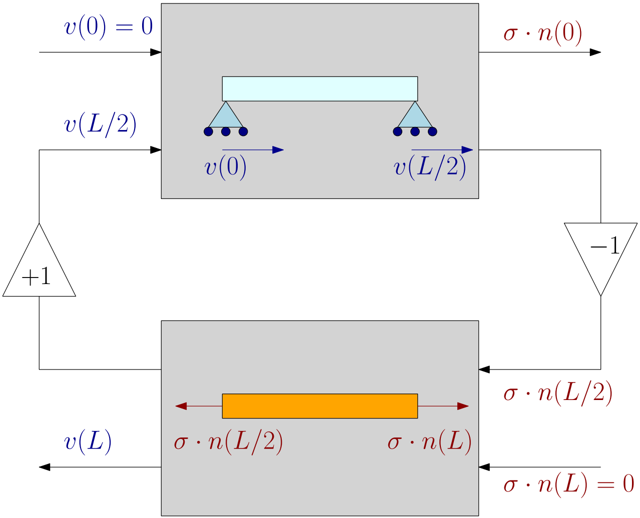


The system can be split into two parts with opposite causalities



Cantilever bar as two interconnected systems

The cantilever bar is then obtained by interconnection



Algebraic interconnection

The left part (l) is described by the dual formulation (Dirichlet bcs)

$$\begin{aligned}\mathbf{M}_{\mathrm{l}}\dot{\mathbf{x}}_{\mathrm{l}} &= \mathbf{J}_{\mathrm{l}}\mathbf{x}_{\mathrm{l}} + \mathbf{B}_{\mathrm{l}}u,\\ y &= \mathbf{B}_{\mathrm{l}}^{\top}\mathbf{x}_{\mathrm{l}}.\end{aligned}$$

The right part (r) is described by the primal formulation (Neumann bcs)

$$\begin{aligned}\mathbf{M}_{\mathrm{r}}\dot{\mathbf{x}}_{\mathrm{r}} &= \mathbf{J}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}} + \mathbf{B}_{\mathrm{r}}u,\\ y &= \mathbf{B}_{\mathrm{r}}^{\top}\mathbf{x}_{\mathrm{r}}.\end{aligned}$$

The interconnection is essentially Newton's third law

$$\begin{aligned}u &= y, \qquad &&\text{The velocity is the same,}\\ u &= -y, \qquad &&\text{The forces are opposite.}\end{aligned}$$

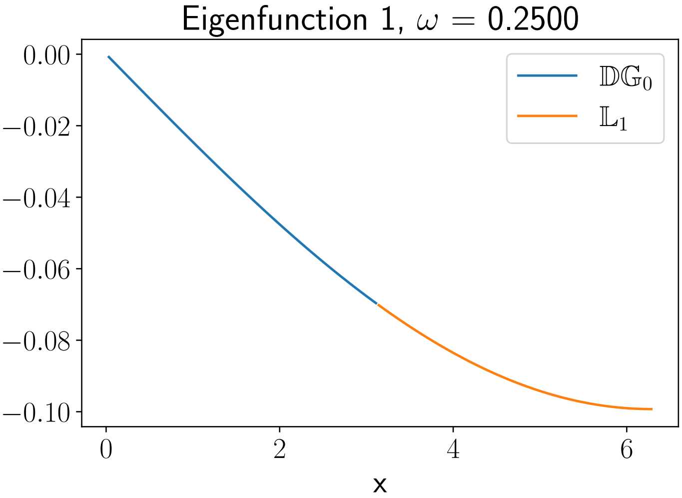
Interconnected system

The interconnected system can be written as follows

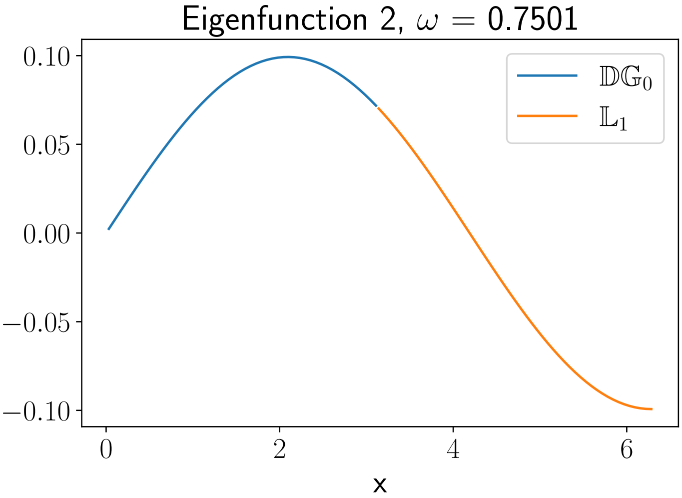
$$\begin{bmatrix} \mathbf{M}_\mathrm{l} & 0 \\ 0 & \mathbf{M}_\mathrm{r} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{x}}_l \\ \dot{\mathbf{x}}_r \end{pmatrix} = \begin{bmatrix} \mathbf{J}_\mathrm{l} & +\mathbf{B}_\mathrm{l}\mathbf{B}_\mathrm{r}^\top \\ -\mathbf{B}_\mathrm{r}\mathbf{B}_\mathrm{l}^\top & \mathbf{J}_\mathrm{r} \end{bmatrix} \begin{pmatrix} \mathbf{x}_l \\ \mathbf{x}_r \end{pmatrix}.$$

All the boundary conditions are weakly enforced.

Example: eigenvalues of clamped-free longitudinal bar



Example: eigenvalues of clamped-free longitudinal bar



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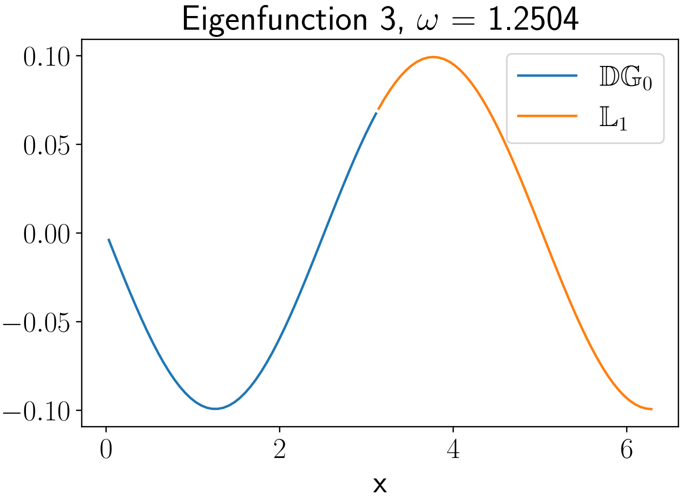
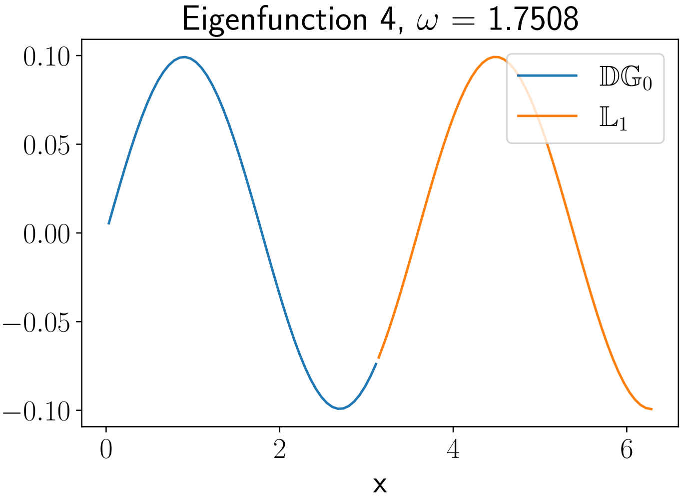


Example: eigenvalues of clamped-free longitudinal bar



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Multidimensional wave equation

$$\begin{bmatrix} c & 0 \\ 0 & \rho \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\sigma} \\ v \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ v \end{pmatrix}.$$

Input and output are now infinite dimensional.

- Neumann control $u_N = \boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\partial\Omega}, \quad y_D = v|_{\partial\Omega}$.
- Dirichlet control $u_D = v|_{\partial\Omega}, \quad y_N = \boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\partial\Omega}$.

In higher space dimensions, the weak two formulations are **not symmetrical anymore**.

Primal and dual weak formulations

Neumann control

Find $\boldsymbol{\sigma} \in L^2(\Omega, \mathbb{R}^d),\ v \in H^1(\Omega)$ such that

$$\begin{aligned}
(\boldsymbol{\xi}_\sigma,\, c\partial_t \boldsymbol{\sigma})_\Omega &= +(\boldsymbol{\xi}_\sigma,\, \operatorname{grad} v)_\Omega, && \forall \boldsymbol{\xi}_\sigma \in L^2(\Omega, \mathbb{R}^d),\\
(\xi_v,\, \rho\partial_t v)_\Omega &= -(\operatorname{grad} \xi_v,\, \boldsymbol{\sigma})_\Omega + (\xi_v,\, u_N)_{\partial\Omega}, && \forall \xi_v \in H^1(\Omega),\\
y_D &= v|_{\partial\Omega},,
\end{aligned}$$

Dirichlet control

Find $\boldsymbol{\sigma} \in H^{\mathrm{div}}(\Omega),\ v \in L^2(\Omega)$ such that

$$\begin{aligned}
(\boldsymbol{\xi}_\sigma,\, c\partial_t \boldsymbol{\sigma})_\Omega &= -(\operatorname{div} \boldsymbol{\xi}_\sigma,\, v)_\Omega + (\boldsymbol{\xi}_\sigma \cdot \boldsymbol{n},\, u_D)_{\partial\Omega}, && \forall \boldsymbol{\xi}_\sigma \in H^{\mathrm{div}}(\Omega, \mathbb{R}^d),\\
(\xi_v,\, \rho\partial_t v)_\Omega &= +(\xi_v,\, \operatorname{div} \boldsymbol{\sigma})_\Omega, && \forall \xi_p \in L^2(\Omega),\\
y_N &= \boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\partial\Omega},
\end{aligned}$$

Choice of the finite element basis (Neumann control)

Neumann control: $(\boldsymbol{\xi}_\sigma,\, c\,\partial_t\boldsymbol{\sigma})_\Omega = (\boldsymbol{\xi}_\sigma,\, \operatorname{grad} v)_\Omega, \qquad \operatorname{grad}\mathcal{V}_h \subset \mathcal{S}_h.$

Choice of the finite element basis (Neumann control)

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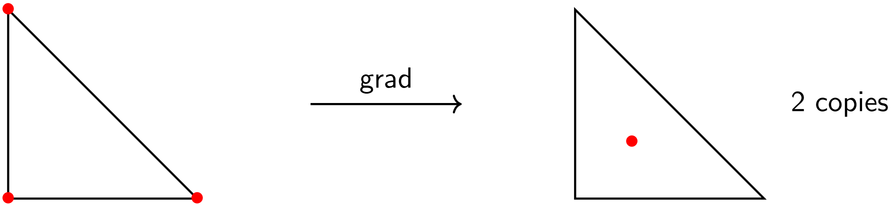


$\mathbb{L}_1$-element:

- $K =$ triangle,
- $P_K := \{a_0 + a_1 x + a_2 y\}$,
- $\Sigma_K :=$ {evaluation on vertices}.

$\mathbb{DG}_0$-element:

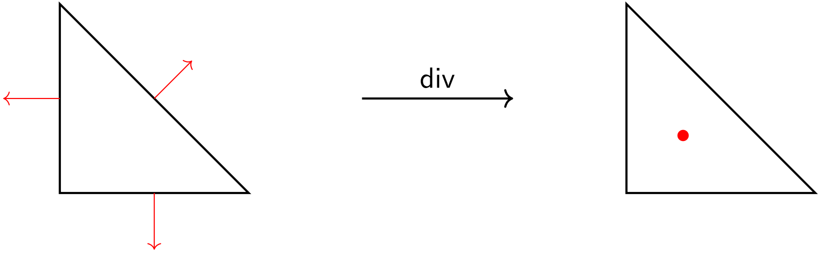
- $K =$ triangle,
- $P_K := \{a_0\}$,
- $\Sigma_K :=$ {evaluation on centroid}.

Choice of the finite element basis (Dirichlet control)

Dirichlet control: $(\xi_v,\, \rho\, \partial_t v)_\Omega = (\xi_v,\, \operatorname{div} \boldsymbol{\sigma})_\Omega, \qquad \operatorname{div} \mathcal{S}_h \subset \mathcal{V}_h.$

Choice of the finite element basis (Dirichlet control)

Dirichlet control: $(\xi_v,\, \rho\,\partial_t v)_\Omega = (\xi_v,\, \operatorname{div} \boldsymbol{\sigma})_\Omega, \qquad \operatorname{div} \mathcal{S}_h \subset \mathcal{V}_h.$



$\mathbb{RT}_0$ (Raviart Thomas)-element:

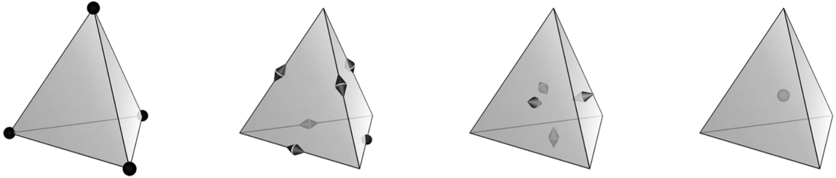
- $K =$ triangle,
- $P_K := \left\{ \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + a_2 \begin{pmatrix} x \\ y \end{pmatrix} \right\}$,
- $\Sigma_K :=$ {integrals over faces}.

$\mathbb{DG}_0$-element:

- $K =$ triangle,
- $P_K := \{a_0\}$,
- $\Sigma_K :=$ {evaluation on centroid}.

Finite element exterior calculus

To obtain stable formulations, finite element exterior calculus can be used[2].

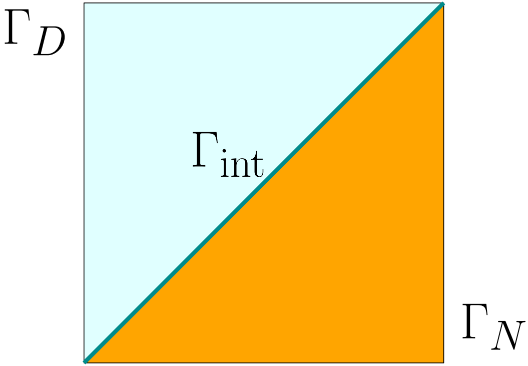


The Whitney forms (1957).

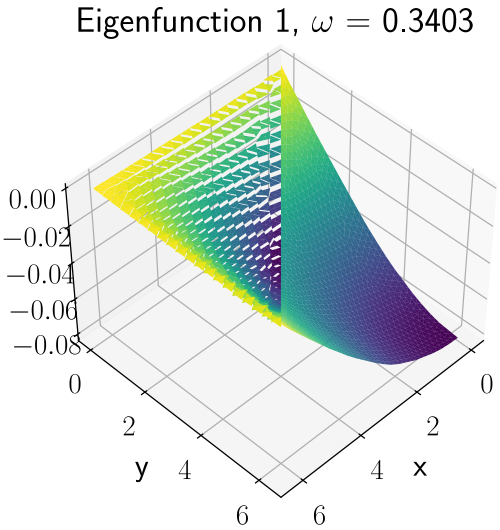
- connection with differential geometry (coordinate free treatment);
- unifying framework for physics;
- clear separation of topological and metrical operations.

[2] Brugnoli, Rashad, and Stramigioli, "Dual field structure-preserving discretization of port-Hamiltonian systems using finite element exterior calculus".

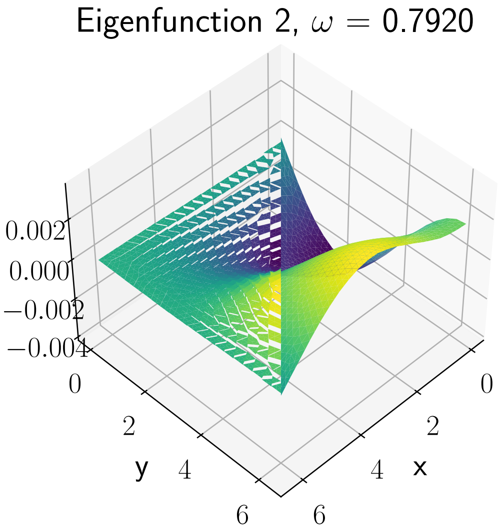
Domain decomposition for two dimensional wave equation



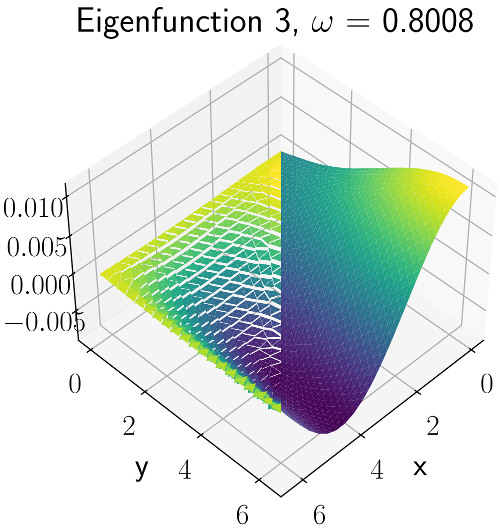
Example: eigenvalues of 2D wave equation



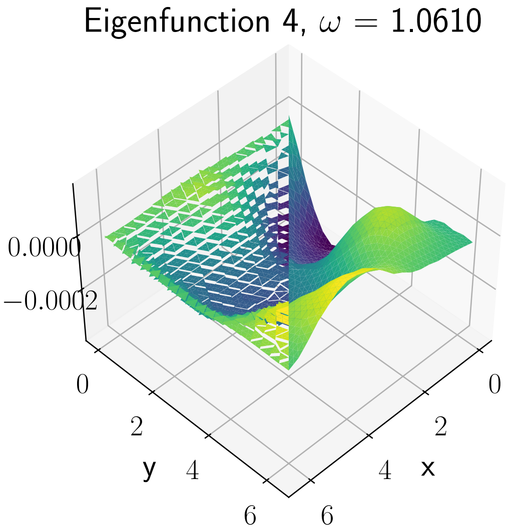
Example: eigenvalues of 2D wave equation



Example: eigenvalues of 2D wave equation



Example: eigenvalues of 2D wave equation



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